CS 361, Lecture 11

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Outline

- Annihilators review
- The final "Lookup Table"
- Non-homogeneous Recurrences
- Limitations

Annihilator Operators

We define three basic operations we can perform on this sequence:

1. Multiply the sequence by a constant: $cA = \langle ca_0, ca_1, ca_2, \cdots \rangle$
2. Shift the sequence to the left: $\mathbf{L}A = \langle a_1, a_2, a_3, \cdots \rangle$
3. Add two sequences: if $A = \langle a_0, a_1, a_2, \cdots \rangle$ and $B = \langle b_0, b_1, b_2, \cdots \rangle$, then $A + B = \langle a_0 + b_0, a_1 + b_1, a_2 + b_2, \cdots \rangle$

Annihilator Description

- We first express our recurrence as a sequence T
- *We use these three operators to "annihilate"* T*, i.e. make it all* 0*'s*
- Key rule: can't multiply by the constant 0
- We can then determine the solution to the recurrence from the sequence of operations performed to annihilate T

Lookup Table

- The annihilator $\mathbf{L} - a$ annihilates sequences of the form $\langle c_1 a^n \rangle$
- The annihilator $(\mathbf{L} - a)(\mathbf{L} - b)$ (where $a \neq b$) anihilates sequences of the form $\langle c_1 a^n + c_2 b^n \rangle$

A Problem

- Our lookup table has a big gap: What does $(\mathbf{L} - a)(\mathbf{L} - a)$ annihilate?
- It turns out it annihilates sequences such as $\langle n a^n \rangle$

Example

$$\begin{aligned}(\mathbf{L}-a)\langle na^n\rangle &= \langle (n+1)a^{n+1} - (a)na^n\rangle \\ &= \langle (n+1)a^{n+1} - na^{n+1}\rangle \\ &= \langle (n+1-n)a^{n+1}\rangle \\ &= \langle a^{n+1}\rangle \\ (\mathbf{L}-a)^2\langle na^n\rangle &= (\mathbf{L}-a)\langle a^{n+1}\rangle \\ &= \langle 0\rangle\end{aligned}$$

Generalization

- It turns out that $(\mathbf{L}-a)^d$ annihilates sequences of the form $\langle p(n)a^n\rangle$ where $p(n)$ is any polynomial of degree $d-1$
- Example: $(\mathbf{L}-1)^3$ annihilates the sequence $\langle n^2 * 1^n\rangle = \langle 1, 4, 9, 16, 25\rangle$ since $p(n) = n^2$ is a polynomial of degree $d-1 = 2$

Lookup Table

- $(\mathbf{L}-a)$ annihilates only all sequences of the form $\langle c_0 a^n\rangle$
- $(\mathbf{L}-a)(\mathbf{L}-b)$ annihilates only all sequences of the form $\langle c_0 a^n + c_1 b^n\rangle$
- $(\mathbf{L}-a_0)(\mathbf{L}-a_1)\ldots(\mathbf{L}-a_k)$ annihilates only sequences of the form $\langle c_0 a_0^n + c_1 a_1^n + \ldots c_k a_k^n\rangle$, here $a_i \neq a_j$, when $i \neq j$
- $(\mathbf{L}-a)^2$ annihilates only sequences of the form $\langle (c_0 n + c_1)a^n\rangle$
- $(\mathbf{L}-a)^k$ annihilates only sequences of the form $\langle p(n)a^n\rangle$, $degree(p(n)) = k-1$

Lookup Table

$$(\mathbf{L}-a_0)^{b_0}(\mathbf{L}-a_1)^{b_1}\ldots(\mathbf{L}-a_k)^{b_k}$$

annihilates only sequences of the form:

$$\langle p_1(n)a_0^n + p_2(n)a_1^n + \ldots p_k(n)a_k^n\rangle$$

where $p_i(n)$ is a polynomial of degree $b_i - 1$ (and $a_i \neq a_j$, when $i \neq j$)

Examples

- Q: What does $(\mathbf{L}-3)(\mathbf{L}-2)(\mathbf{L}-1)$ annihilate?
- A: $c_0 1^n + c_1 2^n + c_2 3^n$
- Q: What does $(\mathbf{L}-3)^2(\mathbf{L}-2)(\mathbf{L}-1)$ annihilate?
- A: $c_0 1^n + c_1 2^n + (c_2 n + c_3)3^n$
- Q: What does $(\mathbf{L}-1)^4$ annihilate?
- A: $(c_0 n^3 + c_1 n^2 + c_2 n + c_3)1^n$
- Q: What does $(\mathbf{L}-1)^3(\mathbf{L}-2)^2$ annihilate?
- A: $(c_0 n^2 + c_1 n + c_2)1^n + (c_3 n + c_4)2^n$

Annihilator Method

- Write down the annihilator for the recurrence
- Factor the annihilator
- Look up the factored annihilator in the "Lookup Table" to get general solution
- Solve for constants of the general solution by using initial conditions

Example

Consider the recurrence $T(n) = 7T(n-1) - 16T(n-2) + 12T(n-3)$, $T(0) = 1$, $T(1) = 5$, $T(2) = 17$

- **Write down the annihilator**: From the definition of the sequence, we can see that $\mathbf{L}^3T - 7\mathbf{L}^2T + 16\mathbf{L}T - 12T = 0$, so the annihilator is $\mathbf{L}^3 - 7\mathbf{L}^2 + 16\mathbf{L} - 12$
- **Factor the annihilator**: We can factor by hand or using a computer program to get $\mathbf{L}^3 - 7\mathbf{L}^2 + 16\mathbf{L} - 12 = (\mathbf{L}-2)^2(\mathbf{L}-3)$
- **Look up to get general solution**: The annihilator $(\mathbf{L}-2)^2(\mathbf{L}-3)$ annihilates sequences of the form $\langle (c_0n + c_1)2^n + c_2 3^n \rangle$
- **Solve for constants**: $T(0) = 1 = c_1 + c_2$, $T(1) = 5 = 2c_0 + 2c_1 + 3c_2$, $T(2) = 17 = 8c_0 + 4c_1 + 9c_2$. We've got three equations and three unknowns. Solving by hand, we get that $c_0 = 1$,$c_1 = 0$,$c_2 = 1$. **Thus:** $T(n) = n2^n + 3^n$

Example (II)

Consider the recurrence $T(n) = 2T(n-1) - T(n-2)$, $T(0) = 0$, $T(1) = 1$

- **Write down the annihilator**: From the definition of the sequence, we can see that $\mathbf{L}^2T - 2\mathbf{L}T + T = 0$, so the annihilator is $\mathbf{L}^2 - 2\mathbf{L} + 1$
- **Factor the annihilator**: We can factor by hand or using the quadratic formula to get $\mathbf{L}^2 - 2\mathbf{L} + 1 = (\mathbf{L}-1)^2$
- **Look up to get general solution**: The annihilator $(\mathbf{L}-1)^2$ annihilates sequences of the form $(c_0n + c_1)1^n$
- **Solve for constants**: $T(0) = 0 = c_1$, $T(1) = 1 = c_0 + c_1$, We've got two equations and two unknowns. Solving by hand, we get that $c_0 = 0$,$c_1 = 1$. **Thus:** $T(n) = n$

In Class Exercise

Consider the recurrence $T(n) = 6T(n-1) - 9T(n-2)$, $T(0) = 1$, $T(1) = 6$

- Q1: What is the annihilator of this sequence?
- Q2: What is the factored version of the annihilator?
- Q3: What is the general solution for the recurrence?
- Q4: What are the constants in this general solution?

(Note: You can check that your general solution works for $T(2)$)

Non-homogeneous terms

- Consider a recurrence of the form $T(n) = T(n-1) + T(n-2) + k$ where k is some constant
- The terms in the equation involving T (i.e. $T(n-1)$ and $T(n-2)$) are called the *homogeneous* terms
- The other terms (i.e.k) are called the *non-homogeneous* terms

Example

- In a *height-balanced tree*, the height of two subtrees of any node differ by at most one
- Let $T(n)$ be the smallest number of nodes needed to obtain a height balanced binary tree of height n
- Q: What is a recurrence for $T(n)$?
- A: Divide this into smaller subproblems
  - To get a height-balanced tree of height n, need one subtree of height $n-1$, and one of height $n-2$, plus a root node
  - Thus $T(n) = T(n-1) + T(n-2) + 1$

Example

- Let's solve this recurrence: $T(n) = T(n-1) + T(n-2) + 1$ (Let $T_n = T(n)$, and $T = \langle T_n \rangle$)
- We know that $(\mathbf{L}^2 - \mathbf{L} - 1)$ annihilates the homogeneous terms
- Let's apply it to the entire equation:

$$\begin{aligned}
(\mathbf{L}^2 - \mathbf{L} - 1)\langle T_n \rangle &= \mathbf{L}^2\langle T_n \rangle - \mathbf{L}\langle T_n \rangle - 1\langle T_n \rangle \\
&= \langle T_{n+2} \rangle - \langle T_{n+1} \rangle - \langle T_n \rangle \\
&= \langle T_{n+2} - T_{n+1} - T_n \rangle \\
&= \langle 1, 1, 1, \cdots \rangle
\end{aligned}$$

Example

- This is close to what we want but we still need to annihilate $\langle 1, 1, 1, \cdots \rangle$
- It's easy to see that $\mathbf{L} - 1$ annihilates $\langle 1, 1, 1, \cdots \rangle$
- Thus $(\mathbf{L}^2 - \mathbf{L} - 1)(\mathbf{L} - 1)$ annihilates $T(n) = T(n-1) + T(n-2) + 1$
- When we factor, we get $(\mathbf{L} - \phi)(\mathbf{L} - \hat{\phi})(\mathbf{L} - 1)$, where $\phi = \frac{1+\sqrt{5}}{2}$ and $\hat{\phi} = \frac{1-\sqrt{5}}{2}$.

Lookup

- Looking up $(\mathbf{L} - \phi)(\mathbf{L} - \hat{\phi})(\mathbf{L} - 1)$ in the table
- We get $T(n) = c_1\phi^n + c_2\hat{\phi}^n + c_3 1^n$
- If we plug in the appropriate initial conditions, we can solve for these three constants
- We'll need to get equations for $T(2)$ in addition to $T(0)$ and $T(1)$

General Rule

To find the annihilator for recurrences with non-homogeneous terms, do the following:

- Find the annihilator a_1 for the homogeneous part
- Find the annihilator a_2 for the non-homogeneous part
- The annihilator for the whole recurrence is then $a_1 a_2$

Another Example

- Consider $T(n) = T(n-1) + T(n-2) + 2$.
- The residue is $\langle 2, 2, 2, \cdots \rangle$ and
- The annihilator is still $(\mathbf{L}^2 - \mathbf{L} - 1)(\mathbf{L} - 1)$, but the equation for $T(2)$ changes!

Another Example

- Consider $T(n) = T(n-1) + T(n-2) + 2^n$.
- The residue is $\langle 1, 2, 4, 8, \cdots \rangle$ and
- The annihilator is now $(\mathbf{L}^2 - \mathbf{L} - 1)(\mathbf{L} - 2)$.

Another Example

- Consider $T(n) = T(n-1) + T(n-2) + n$.
- The residue is $\langle 1, 2, 3, 4, \cdots \rangle$
- The annihilator is now $(\mathbf{L}^2 - \mathbf{L} - 1)(\mathbf{L} - 1)^2$.

Another Example

- Consider $T(n) = T(n-1) + T(n-2) + n^2$.
- The residue is $\langle 1, 4, 9, 25, \cdots \rangle$ and
- The annihilator is $(\mathbf{L}^2 - \mathbf{L} - 1)(\mathbf{L} - 1)^3$.

Another Example

- Consider $T(n) = T(n-1) + T(n-2) + n^2 - 2^n$.
- The residue is $\langle 1-1, 4-4, 9-8, 25-16, \cdots \rangle$ and the
- The annihilator is $(\mathbf{L}^2 - \mathbf{L} - 1)(\mathbf{L} - 1)^3(\mathbf{L} - 2)$.

Limitations

- Our method does not work on $T(n) = T(n-1) + \frac{1}{n}$ or $T(n) = T(n-1) + \lg n$
- The problem is that $\frac{1}{n}$ and $\lg n$ do not have annihilators.
- Our tool, as it stands, is limited