CS 361, Lecture 23

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Outline

- Red Black Trees

What is a RB-Tree

- A RB-Tree is a balanced binary search tree
- The height of the tree is always $O(\log n)$ where n is the number of nodes in the tree

RB Trees

- Each node has a "color" field in addition to a key, left, right, and parent pointer
- If the child or parent of a node does not exist, the corresponding pointer field will contain the value NIL
- We will say that these NIL's are pointers to external nodes (leaves) of the tree, and say that all key-bearing nodes are internal nodes of the tree

Red-Black Properties

A BST is a red-black tree if it satisfies the RB-Properties

1. Every node is either red or black
2. The root is black
3. Every leaf (NIL) is black
4. If a node is red, then both its children are black
5. For each node, all paths from the node to descendant leaves contain the same number of black nodes

Example RB-Tree

Black Height

- *Black-height* of a node x, bh(x) is the number of nodes on any path from, but not including x down to a leaf node.
- Note that the black-height of a node is well-defined since all paths have the same number of black nodes
- The black-height of an RB-Tree is just the black-height of the root

Key Lemma

- *Lemma: A RB-Tree with n internal nodes has height at most* $2\log(n+1)$
- Proof Sketch: 1) The subtree rooted at the node x contains at least $2^{bh(x)}-1$ internal nodes
- 2) For the root r, $bh(r) \geq h/2$, thus $n \geq 2^{h/2}-1$. Taking logs of both sides, we get that $h \leq 2\log(n+1)$

Proof

1) The subtree rooted at the node x contains at least $2^{bh(x)}-1$ internal nodes

- Show by induction on the height of x
- BC: If the height of x is 0, then x is a leaf, and subtree rooted at x does indeed contain $2^0-1=0$ internal nodes
- IS: Consider a node x which is an internal node with two children(all internal nodes have two children). Each child has black-height of either $bh(x)$ or $bh(x)-1$ (the former if it is red, the latter if it is black). Since the height of these children is less than x, we can apply the inductive hypothesis to conclude that each child has at least $2^{bh(x)-1}-1$ internal nodes. This implies that the subtree rooted at x has at least $(2^{bh(x)-1}-1)+(2^{bh(x)-1}-1)+1=2^{bh(x)}-1$ internal nodes. This proves the claim.

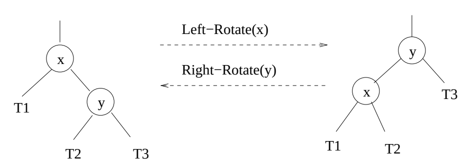
Maintenance?

- How do we ensure that the Red-Black Properties are maintained?
- I.e. when we insert a new node, what do we color it? How do we re-arrange the new tree so that the Red-Black Property holds?
- How about for deletions?

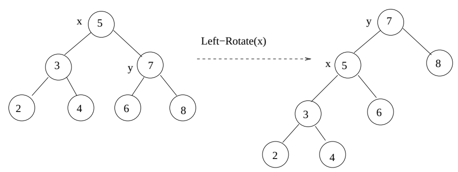
Left-Rotate

- Left-Rotate(x) takes a node x and "rotates" x with its right child
- Right-Rotate is the symmetric operation
- *Both Left-Rotate and Right-Rotate preserve the BST Property*
- We'll use Left-Rotate and Right-Rotate in the RB-Insert procedure

Picture



Example



Binary Search Tree Property

- Let x be a node in a binary search tree. If y is a node in the left subtree of x, then key(y)$\leq$key(x). If y is a node in the right subtree of x then key(y)$\geq$key(x)

In-Class Exercise

Show that Left-Rotate(x) maintains the BST Property. In other words, show that if the BST Property was true for the tree before the Left-Rotate(x) operation, then it's true for the tree after the operation.

- Show that after rotation, the BST property holds for the entire subtree rooted at x
- Show that after rotation, the BST property holds for the subtree rooted at y
- Now argue that after rotation, the BST property holds for the entire tree

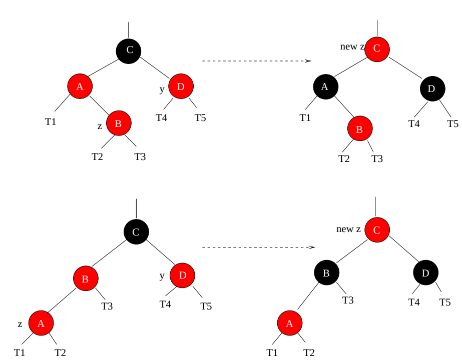
RB-Insert(T,z)

1. Set left(z) and right(z) to be NIL
2. Let y be the last node processed during a search for z in T
3. Insert z as the appropriate child of y (left child if key(z)$\leq$ y, right child otherwise)
4. Color z red
5. Call the procedure RB-Insert-Fixup

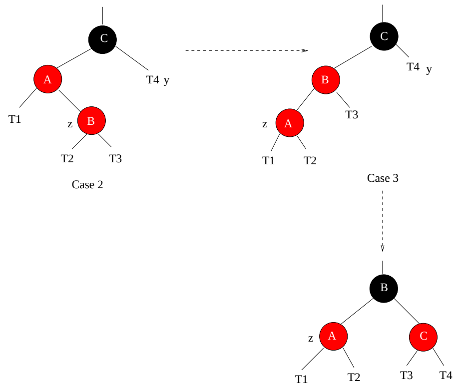
RB-Insert-Fixup(T,z)

```
RB-Insert-Fixup(T,z){
  while (color(p(z)) is red){
    case 1: z's uncle, y, is red{
      do case 1
    }
    case 2: z's uncle, y, is black and z is a right child{
      do case 2
    }
    case 3: z's uncle, y, is black and z is a left child{
      do case 3
    }
  }
  color(root(T)) = black;
}
```


Case 1



Case 2 and 3



Loop Invariant

At the start of each iteration of the loop:

- Node z is red
- If parent(z) is the root, then parent(z) is black
- If there is a violation of the red-black properties, there is at most one violation, and it is either property 2 or 4. If there is a violation of property 2, it occurs because z is the root and is red. If there is a violation of property 4, it occurs because both z and parent(z) are red.

Pseudocode

- Detailed Pseudocode for RB-Insert and RB-Insert-Fixup is in the book, Chapter 13.3
- There's also a detailed proof of correctness for RB-Insert-Fixup in the the same Chapter