

- T(n) = 2 \* T(n/2) + n is an example of a recurrence relation
  A Recurrence Relation is any equation for a function T, where
- T appears on both the left and right sides of the equation.We always want to "solve" these recurrence relation by get-
- ting an equation for T, where T appears on just the left side of the equation
- The next several problems can be attacked by induction/recurrences
- For each problem, we'll need to reduce it to smaller problems
   Question: How can we reduce each problem to a smaller
- Question: How can we reduce each problem to a smaller subproblem?

Sum Problem	Dominoes Problem
• $f(n)$ is the sum of the integers $1, \ldots, n$	• $f(n)$ is the number of ways to tile a 2 by $n$ rectangle with dominoes (a domino is a 2 by 1 rectangle)
6	9
Tree Problem	Simpler Subproblems
<ul> <li>f(n) is the maximum number of leaf nodes in a binary tree of height n</li> <li>Recall:</li> <li>In a binary tree, each node has at most two children</li> <li>A <i>leaf</i> node is a node with no children</li> <li>The height of a tree is the length of the longest path from the root to a leaf node.</li> </ul>	<ul> <li>Sum Problem: What is the sum of all numbers between 1 and n-1 (i.e. f(n-1))?</li> <li>Tree Problem: What is the maximum number of leaf nodes in a binary tree of height n-1? (i.e. f(n-1))</li> <li>Binary Search Problem: What is the maximum number of queries that need to be made for binary search on a sorted array of size n/2? (i.e. f(n/2))</li> <li>Dominoes problem: What is the number of ways to tile a 2 by n - 1 rectangle with dominoes? What is the number of ways to tile a 2 by n - 2 rectangle with dominoes? (i.e. f(n-1), f(n-2))</li> </ul>
7	10
Binary Search Problem	Recurrences
• $f(n)$ is the maximum number of queries that need to be made for binary search on a sorted array of size $n$ .	<ul> <li>Sum Problem: f(n) = f(n - 1) + n, f(1) = 1</li> <li>Tree Problem: f(n) = 2 * f(n - 1), f(0) = 1</li> <li>Binary Search Problem: f(n) = f(n/2) + 1, f(1) = 0</li> <li>Dominoes problem: f(n) = f(n - 1) + f(n - 2), f(1) = 1, f(2) = 1</li> </ul>

Guesses	Tree Problem
• Sum Problem: $f(n) = (n + 1)n/2$ • Tree Problem: $f(n) = 2^n$ • Binary Search Problem: $f(n) = \log n$ • Dominoes problem: $f(n) = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n$	• Want to show that $f(n) = 2^n$ . • Prove by induction on $n$ • Base case: $f(0) = 2^0 = 1$ • Inductive hypothesis: for all $j < n$ , $f(j) = 2^j$ • Inductive step: f(n) = 2 * f(n-1) (4) $= 2 * (2^{n-1})$ (5) $= 2^n$ (6)
12	15
Inductive Proofs	Binary Search Problem
<ul> <li>"Trying is the first step to failure" - Homer Simpson</li> <li>Now that we've made these guesses, we can try using induction to prove they're correct</li> <li>(This is the Substitution Method)</li> </ul>	<ul> <li>Want to show that f(n) = log n. (assume n is a power of 2)</li> <li>Prove by induction on n</li> <li>Base case: f(1) = log 1 = 0</li> <li>Inductive hypothesis: for all j &lt; n, f(j) = log j</li> <li>Inductive step:</li> </ul>
<ul> <li>(This is the Substitution Method)</li> <li>We'll give inductive proofs that these guesses are correct for the first three problems</li> </ul>	f(n) = f(n/2) + 1 (7) $= \log n/2 + 1$ (8) $= \log n - \log 2 + 1$ (9) $= \log n$ (10)
13	16
Sum Problem	In Class Exercise
<ul> <li>Want to show that f(n) = (n + 1)n/2.</li> <li>Prove by induction on n</li> <li>Base case: f(1) = 2 * 1/2 = 1</li> <li>Inductive hypothesis: for all j &lt; n, f(j) = (j + 1)j/2</li> </ul>	Consider the following interview question:

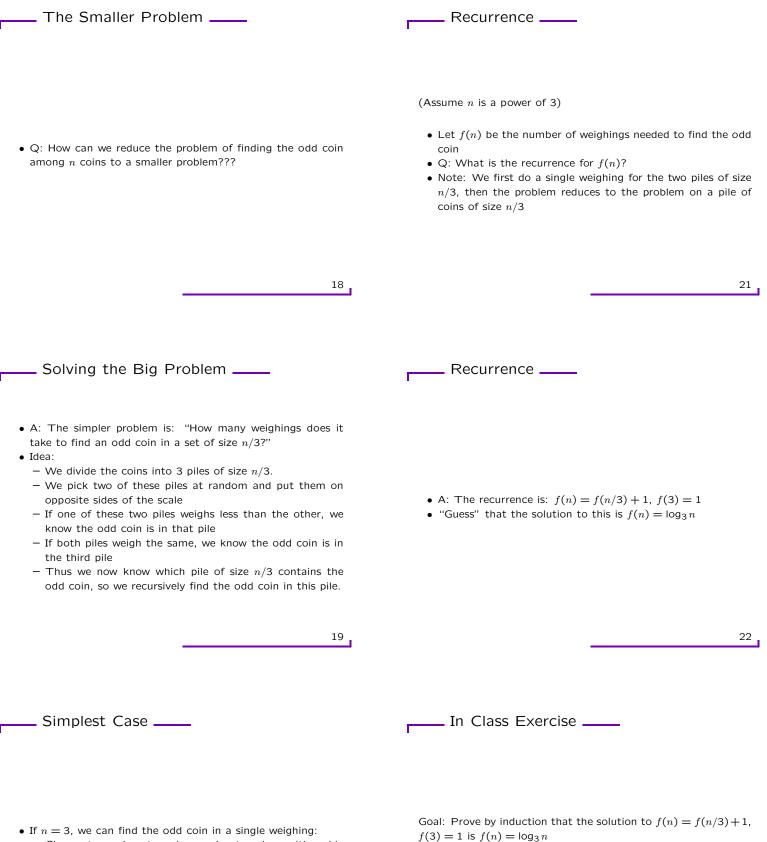
- Inductive hypothesis: for all j < n, f(j) = (j+1)j/2
- Inductive step:

$$f(n) = f(n-1) + n$$
 (1)

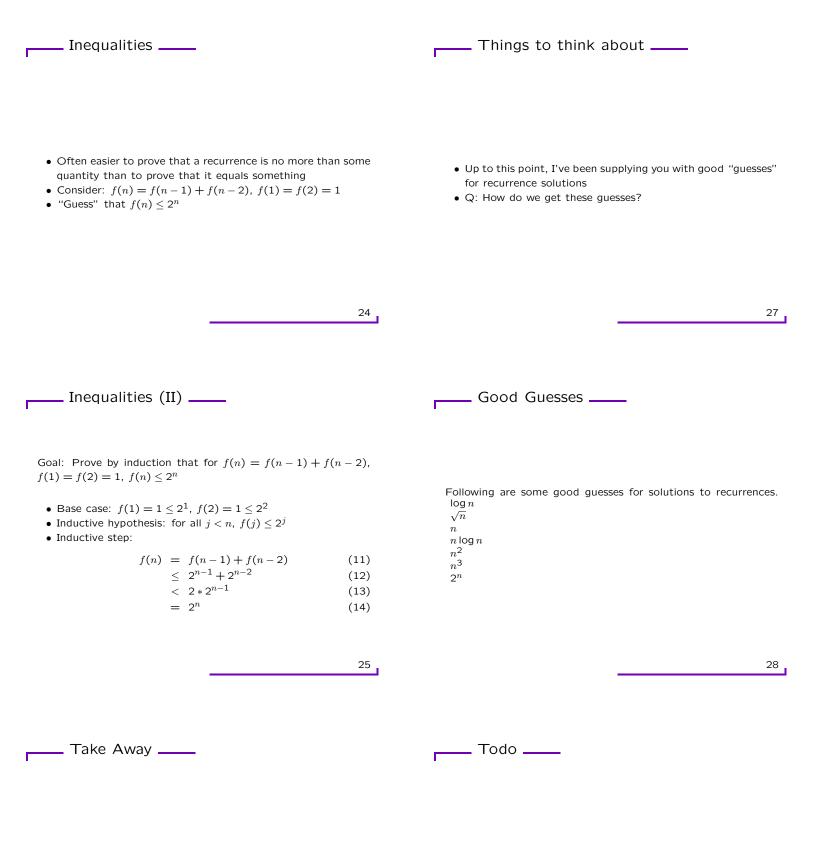
$$= n(n-1)/2 + n$$
 (2)

$$= (n+1)n/2$$
 (3)

- $\bullet$  Out of n coins, one weighs less than the others
- You have a scale
- What is the minimum number of weighs on the scale you can do to find the odd coin?



- Choose two coins at random and put each on either side of the scale
- If both weigh the same, odd coin is the third one. If one coin weighs less, that coin is the odd one
- Q1: What is the base case? Prove that it holds.
- Q2: What is the inductive hypothesis?
- Q3: Prove the inductive step.



- Recurrences and Induction are closely related
- Both techniques require that we solve a big problem by using a solution to a smaller problem
- One technique for solving recurrences is to "guess" the solution and then prove this guess is right by induction
- Read Chapter 4 in book (skip proof of the Masters Theorem)