CS 461, Lecture 10

Jared Saia
University of New Mexico

Today's Outline

- Fractional Knapsack Wrapup
- Midterm Review

Proof

- Assume the objects are sorted in order of cost per pound. Let v_i be the value for item i and let w_i be its weight.
- Let x_i be the *fraction* of object i selected by greedy and let V be the total value obtained by greedy
- Consider some arbitrary solution, B, and let x'_i be the fraction of object i taken in B and let V' be the total profit obtained by B
- We want to show that $V' \leq V$ or that $V - V' \geq 0$

Proof

- Let k be the smallest index with $x_k < 1$
- Note that for $i \leq k$, $x_i = 1$ and for $i > k$, $x_i = 0$
- You will show that for all i,

$$(x_i - x'_i)\frac{v_i}{w_i} \geq (x_i - x'_i)\frac{v_k}{w_k}$$

Proof

$$
\begin{aligned}
V - V' &= \sum_{i=1}^{n} x_i v_i - \sum_{i=1}^{n} x'_i v_i && (1)\\
&= \sum_{i=1}^{n} (x_i - x'_i) * v_i && (2)\\
&= \sum_{i=1}^{n} (x_i - x'_i) * w_i \left(\frac{v_i}{w_i}\right) && (3)\\
&\geq \sum_{i=1}^{n} (x_i - x'_i) * w_i \left(\frac{v_k}{w_k}\right) && (4)\\
&\geq \left(\frac{v_k}{w_k}\right) * \sum_{i=1}^{n} (x_i - x'_i) * w_i && (5)\\
&\geq 0 && (6)
\end{aligned}
$$

Proof

- Note that the last step follows because $\frac{v_k}{w_k}$ is positive and because:

$$
\begin{aligned}
\sum_{i=1}^{n} (x_i - x'_i) * w_i &= \sum_{i=1}^{n} x_i w_i - \sum_{i=1}^{n} x'_i * w_i && (7)\\
&= W - W' && (8)\\
&\geq 0. && (9)
\end{aligned}
$$

- Where W is the total weight taken by greedy and W' is the total weight for the strategy B
- We know that $W \geq W'$

In-Class Exercise

Consider the inequality:

$$(x_i - x'_i)\frac{v_i}{w_i} \geq (x_i - x'_i)\frac{v_k}{w_k}$$

- Q1: Show this inequality is true for $i < k$
- Q2: Show it's true for $i = k$
- Q3: Show it's true for $i > k$

Q1

$$(x_i - x'_i)\frac{v_i}{w_i} \geq (x_i - x'_i)\frac{v_k}{w_k}$$

- Q1: Show that the inequality is true for $i < k$
- For $i < k$, $(x_i - x'_i) \geq 0$
- If $(x_i - x'_i) = 0$, trivially true. Otherwise, can divide both sides of the inequality by $x_i - x'_i$ to get

$$\frac{v_i}{w_i} \geq \frac{v_k}{w_k}.$$

- This is true since the items are sorted by profit per weight

Q2

$$(x_i - x_i')\frac{v_i}{w_i} \geq (x_i - x_i')\frac{v_k}{w_k}$$

- Q2: Show that the inequality is true for $i = k$
- When $i = k$, we have

$$(x_k - x_k')\frac{v_k}{w_k} \geq (x_k - x_k')\frac{v_k}{w_k}$$

- Which is true since the left side equals the right side

Q3

$$(x_i - x_i')\frac{v_i}{w_i} \geq (x_i - x_i')\frac{v_k}{w_k}$$

- Q3: Show that the inequality is true for $i > k$
- For $i < k$, $(x_i - x_i') \leq 0$
- If $(x_i - x_i') = 0$, trivially true. Otherwise can divide both sides of the inequality by $x_i - x_i'$ to get

$$\frac{v_i}{w_i} \leq \frac{v_k}{w_k}.$$

- This is obviously true since the items are sorted by profit per weight
- *Note that the direction of the inequality changed when we divided by $(x_i - x_i')$, since it is negative*

Midterm Info

- Midterm will be Tuesday, Sept. 30th at regular class time and place
- You can bring 2 pages of "cheat sheets" to use during the exam. You can also bring a calculator. Otherwise the exam is closed book and closed note.
- Note that the web page contains links to prior classes and their midterms. *Many of the questions on my midterm will be similar in flavor to these past midterms!*

Midterm Review Session

- I will have a review session Monday, Sept 29th at 5:30pm in FEC 141 (the conference room on the first floor of FEC)
- Maxwell will also have a review session
- Please come with questions

Midterm

- 5 questions, about 20 points each
- Hard but fair
- There will be some time pressure, so make sure you can e.g. solve recurrences both quickly and correctly.
- I expect a class mean of between 60 :(and 70 :) points

Question 1

Collection of true/false questions and short answer on:

- Asymptotic notation: e.g. I give you a bunch of functions and ask you to give me the simplest possible theta notation for each
- Recurrences: e.g. I ask you to solve a recurrence
- Dynamic Programming: general concepts, string alignment, matrix multiplication shortest common subsequence (know resource bounds for these algorithms)
- Greedy Algorithms: general concepts, activity selection, fractional knapsack (know resource bounds for these algorithms)

Question 2

- A question on recurrence relations
- like problems 1,2,3 of hw 2
- You'll need to know annihilators, change of variables, handling homogeneous and non-homogeneous parts of recurrences, recursion trees, and the Master Method
- You'll need to know the formulas for sums of convergent and divergent geometric series

Question 3

- Asymptotic notation
- Similar to hw problem 3.1-2, 3.1-5, 3.1-7

Question 4

- Recurrence proof using induction (i.e. the susbstitution method)
- You'll need to give base case, inductive hypothesis and then show the inductive step
- Similar to Exercise 15.2-3, Exercise 4.2-1 and Exercise 4.2-3

Questions 5

- Question on Dynamic Programming
- Will ask you to solve some problem using Dynamic Programming
- Will likely be some variant on one of the following problems: string alignment, matrix multiplication, longest common subsequence or the monotonically increasing subsequence problem of Exercise 15.4-5 from hw2

Asymptotic Notation

- Let's now review asymptotic notation
- I'll review for O notation, make sure you understand the other four types
- $f(n) = O(g(n))$ *if there exists positive constants* c *and* n_0 *such that* $0 \leq f(n) \leq cg(n)$ *for all* $n \geq n_0$
- This means to show that $f(n) = O(g(n))$, *you need to give positive constants* c *and* n_0 for which the above statement is true!

Example 1

- Prove that $2^{n+1} = O(2^n)$
- Goal: Show there exist positive constants c and n_0 such that $2^{n+1} \leq c * 2^n$ for all $n \geq n_0$

$$
\begin{aligned}
2^{n+1} &\leq c * 2^n && (10)\\
2 * 2^n &\leq c * 2^n && (11)\\
2 &\leq c && (12)
\end{aligned}
$$

- Hence for $c = 2$ and $n_0 = 1$, $2^{n+1} \leq c * 2^n$ for all $n \geq n_0$

Example 2

- Prove that $n + \sqrt{n} = O(n)$
- Goal: Show there exist positive constants c and n_0 such that $n + \sqrt{n} \leq c * n$ for all $n \geq n_0$

$$
\begin{aligned}
n + \sqrt{n} &\leq c * n && (13)\\
1 + \frac{1}{\sqrt{n}} &\leq c && (14)\\
& && (15)
\end{aligned}
$$

- Hence if we choose $n_0 = 4$, and $c = 1.5$, then it's true that $n + \sqrt{n} \leq c * n$ for all $n \geq n_0$

Example 3

- Prove that $2^{2n} = O(5^n)$
- Goal: Show there exist positive constants c and n_0 such that $2^{2n} \leq c * 5^n$ for all $n \geq n_0$

$$
\begin{aligned}
2^{2n} &\leq c * 5^n && (16)\\
4^n &\leq c * 5^n && (17)\\
(4/5)^n &\leq c && (18)\\
& && (19)
\end{aligned}
$$

- Hence for $c = 1$ and $n_0 = 1$, $2^{2n} \leq c * 5^n$ for all $n \geq n_0$

A Procedure

Goal: prove that $f(n) = O(g(n))$

1. Write down what this means mathematically
2. Write down the inequality $f(n) \leq c * g(n)$
3. Simplify this inequality so that c is isolated on the right hand side
4. Now find a n_0 and a c such that for all $n \geq n_0$, this simplified inequality is true

In Class Exercise

Show that $n2^n$ is $O(4^n)$

- Q1: What is the exact mathematical statement of what you need to prove?
- Q2: What is the first inequality in the chain of inequalities?
- Q3: What is the simplified inequality where c is isolated?
- Q4: What is a n_0 and c such that the inequality of the last question is always true?