Annihilator Method _____

CS 461, Lecture 4

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- Write down the annihilator for the recurrence
- Factor the annihilator
- Look up the factored annihilator in the "Lookup Table" to get general solution
- Solve for constants of the general solution by using initial conditions

_ Today's Outline ____

____ Lookup Table ____

• Annihilators for recurrences with non-homogeneous terms

• Transformations

$$(\mathbf{L} - a_0)^{b_0} (\mathbf{L} - a_1)^{b_1} \dots (\mathbf{L} - a_k)^{b_k}$$

annihilates only sequences of the form:

$$\langle p_1(n)a_0^n + p_2(n)a_1^n + \dots p_k(n)a_k^n \rangle$$

where $p_i(n)$ is a polynomial of degree $b_i - 1$ (and $a_i \neq a_j$, when $i \neq j$)

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• Q: What does (L - 3)(L - 2)(L - 1) annihilate?

- A: $c_0 1^n + c_1 2^n + c_2 3^n$
- Q: What does $(L-3)^2(L-2)(L-1)$ annihilate?
- A: $c_0 1^n + c_1 2^n + (c_2 n + c_3) 3^n$
- Q: What does $(L-1)^4$ annihilate?
- A: $(c_0n^3 + c_1n^2 + c_2n + c_3)1^n$
- Q: What does $(L 1)^3(L 2)^2$ annihilate?
- A: $(c_0n^2 + c_1n + c_2)1^n + (c_3n + c_4)2^n$

Example (II) _____

Consider the recurrence T(n) = 2T(n-1) - T(n-2), T(0) = 0, T(1) = 1

- Write down the annihilator: From the definition of the sequence, we can see that $L^2T-2LT+T = 0$, so the annihilator is $L^2 2L + 1$
- Factor the annihilator: We can factor by hand or using the quadratic formula to get $L^2 2L + 1 = (L 1)^2$
- Look up to get general solution: The annihilator $(L-1)^2$ annihilates sequences of the form $(c_0n + c_1)1^n$
- Solve for constants: $T(0) = 0 = c_1$, $T(1) = 1 = c_0 + c_1$, We've got two equations and two unknowns. Solving by hand, we get that $c_0 = 0, c_1 = 1$. Thus: T(n) = n

Example _____

Consider the recurrence T(n) = 7T(n-1) - 16T(n-2) + 12T(n-3), T(0) = 1, T(1) = 5, T(2) = 17

- Write down the annihilator: From the definition of the sequence, we can see that $L^{3}T 7L^{2}T + 16LT 12T = 0$, so the annihilator is $L^{3} 7L^{2} + 16L 12$
- Factor the annihilator: We can factor by hand or using a computer program to get $L^3-7L^2+16L-12 = (L-2)^2(L-3)$
- Look up to get general solution: The annihilator $(L 2)^2(L 3)$ annihilates sequences of the form $\langle (c_0n + c_1)2^n + c_23^n \rangle$
- Solve for constants: $T(0) = 1 = c_1 + c_2$, $T(1) = 5 = 2c_0 + 2c_1 + 3c_2$, $T(2) = 17 = 8c_0 + 4c_1 + 9c_2$. We've got three equations and three unknowns. Solving by hand, we get that $c_0 = 1, c_1 = 0, c_2 = 1$. Thus: $T(n) = n2^n + 3^n$

Consider the recurrence T(n) = 6T(n-1) - 9T(n-2), T(0) = 1, T(1) = 6

• Q1: What is the annihilator of this sequence?

At Home Exercise _____

- Q2: What is the factored version of the annihilator?
- Q3: What is the general solution for the recurrence?
- Q4: What are the constants in this general solution?

(Note: You can check that your general solution works for T(2))

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Non-homogeneous terms _____

__ Example ____

- Consider a recurrence of the form T(n) = T(n-1) + T(n-2) + k where k is some constant
- The terms in the equation involving T (i.e. T(n-1) and T(n-2)) are called the *homogeneous* terms
- The other terms (i.e.k) are called the *non-homogeneous* terms

- Let's solve this recurrence: T(n) = T(n-1) + T(n-2) + 1(Let $T_n = T(n)$, and $T = \langle T_n \rangle$)
- We know that (L^2-L-1) annihilates the homogeneous terms
- Let's apply it to the entire equation:

$$(\mathbf{L}^{2} - \mathbf{L} - 1)\langle T_{n} \rangle = \mathbf{L}^{2} \langle T_{n} \rangle - \mathbf{L} \langle T_{n} \rangle - 1 \langle T_{n} \rangle$$

= $\langle T_{n+2} \rangle - \langle T_{n+1} \rangle - \langle T_{n} \rangle$
= $\langle T_{n+2} - T_{n+1} - T_{n} \rangle$
= $\langle 1, 1, 1, \dots \rangle$

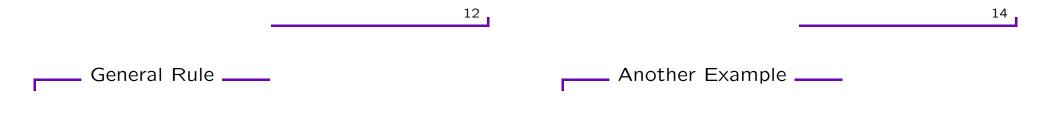


- In a *height-balanced tree*, the height of two subtrees of any node differ by at most one
- Let T(n) be the smallest number of nodes needed to obtain a height balanced binary tree of height n
- Q: What is a recurrence for T(n)?
- A: Divide this into smaller subproblems
 - To get a height-balanced tree of height n with the smallest number of nodes, need one subtree of height n 1, and one of height n 2, plus a root node
 - Thus T(n) = T(n-1) + T(n-2) + 1

- This is close to what we want but we still need to annihilate $\langle 1,1,1,\cdots\rangle$
- \bullet It's easy to see that L-1 annihilates $\langle 1,1,1,\cdots\rangle$
- Thus $(L^2 L 1)(L 1)$ annihilates T(n) = T(n 1) + T(n 2) + 1
- When we factor, we get $(\mathbf{L}-\phi)(\mathbf{L}-\hat{\phi})(\mathbf{L}-1)$, where $\phi = \frac{1+\sqrt{5}}{2}$ and $\hat{\phi} = \frac{1-\sqrt{5}}{2}$.

- Looking up $(\mathbf{L}-\phi)(\mathbf{L}-\hat{\phi})(\mathbf{L}-1)$ in the table
- We get $T(n) = c_1 \phi^n + c_2 \hat{\phi}^n + c_3 \mathbf{1}^n$
- If we plug in the appropriate initial conditions, we can solve for these three constants
- We'll need to get equations for T(2) in addition to T(0) and T(1)

- Consider T(n) = T(n-1) + T(n-2) + 2.
- \bullet The residue is $\langle 2,2,2,\cdots\rangle$ and
- The annihilator is still $(L^2 L 1)(L 1)$, but the equation for T(2) changes!



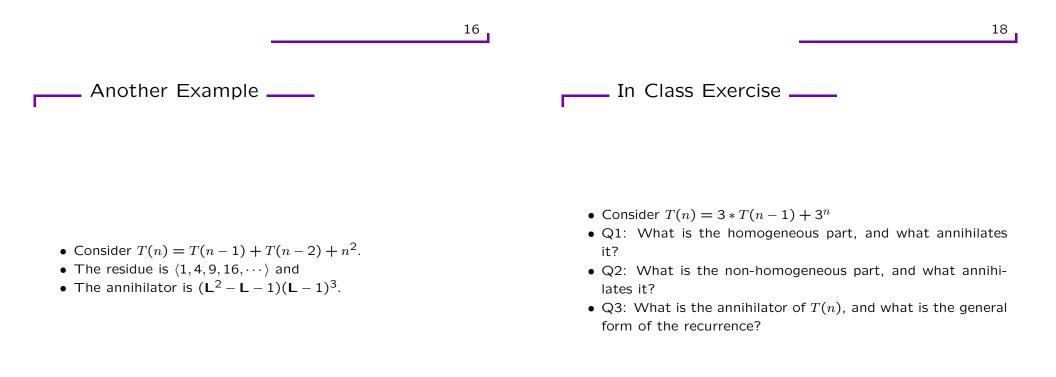
To find the annihilator for recurrences with non-homogeneous terms, do the following:

- Find the annihilator a_1 for the homogeneous part
- Find the annihilator a_2 for the non-homogeneous part
- The annihilator for the whole recurrence is then a_1a_2

- Consider $T(n) = T(n-1) + T(n-2) + 2^n$.
- The residue is $\langle 1,2,4,8,\cdots\rangle$ and
- The annihilator is now $(\mathbf{L}^2 \mathbf{L} 1)(\mathbf{L} 2)$.

- Consider T(n) = T(n-1) + T(n-2) + n.
- The residue is $\langle 1, 2, 3, 4, \cdots \rangle$
- The annihilator is now $(L^2 L 1)(L 1)^2$.

- Consider $T(n) = T(n-1) + T(n-2) + n^2 2^n$.
- The residue is $\langle 1-1, 4-4, 9-8, 16-16, \cdots \rangle$ and the
- The annihilator is $(L^2 L 1)(L 1)^3(L 2)$.

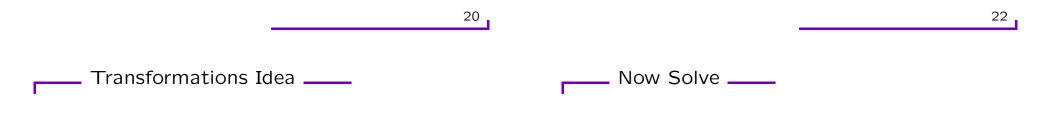


Limitations _____

— Transformation ——

- Our method does not work on $T(n) = T(n-1) + \frac{1}{n}$ or $T(n) = T(n-1) + \lg n$
- The problem is that $\frac{1}{n}$ and $\lg n$ do not have annihilators.
- Our tool, as it stands, is limited.
- Key idea for strengthening it is *transformations*

- Let $n = 2^i$ and rewrite T(n):
- $T(2^0) = 1$ and $T(2^i) = 2T(\frac{2^i}{2}) + k2^i = 2T(2^{i-1}) + k2^i$
- Now define a new sequence t as follows: $t(i) = T(2^i)$
- Then t(0) = 1, $t(i) = 2t(i-1) + k2^i$



- Consider the recurrence giving the run time of mergesort T(n) = 2T(n/2) + kn (for some constant k), T(1) = 1
- How do we solve this?
- We have no technique for annihilating terms like T(n/2)
- However, we can *transform* the recurrence into one with which we can work

- We've got a new recurrence: t(0) = 1, $t(i) = 2t(i-1) + k2^i$
- We can easily find the annihilator for this recurrence
- (L − 2) annihilates the homogeneous part, (L − 2) annihilates the non-homogeneous part, So (L − 2)(L − 2) annihilates t(i)
- Thus $t(i) = (c_1i + c_2)2^i$

Reverse Transformation _____

- We've got a solution for t(i) and we want to transform this into a solution for T(n)
- Recall that $t(i) = T(2^i)$ and $2^i = n$

$$t(i) = (c_1 i + c_2)2^i \tag{1}$$

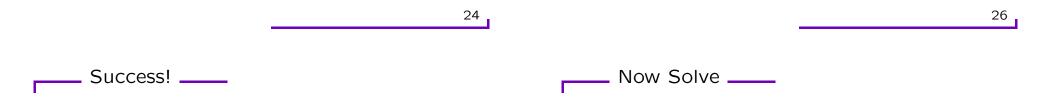
$$T(2^{i}) = (c_{1}i + c_{2})2^{i}$$
(2)

$$T(n) = (c_1 \lg n + c_2)n$$
 (3)

$$= c_1 n \lg n + c_2 n \tag{4}$$

$$= \Theta(n \lg n) \tag{5}$$

- Consider the recurrence $T(n) = 9T(\frac{n}{3}) + kn$, where T(1) = 1and k is some constant
- Let $n = 3^i$ and rewrite T(n):
- $T(2^0) = 1$ and $T(3^i) = 9T(3^{i-1}) + k3^i$
- Now define a sequence t as follows $t(i) = T(3^i)$
- Then t(0) = 1, $t(i) = 9t(i-1) + k3^i$



Let's recap what just happened:

- We could not find the annihilator of T(n) so:
- We did a *transformation* to a recurrence we could solve, t(i) (we let n = 2ⁱ and t(i) = T(2ⁱ))
- We found the annihilator for t(i), and solved the recurrence for t(i)
- We reverse transformed the solution for t(i) back to a solution for T(n)

- $t(0) = 1, t(i) = 9t(i-1) + k3^i$
- This is annihilated by (L 9)(L 3)
- So t(i) is of the form $t(i) = c_19^i + c_23^i$

- $t(i) = c_1 9^i + c_2 3^i$
- Recall: $t(i) = T(3^i)$ and $3^i = n$

$$t(i) = c_1 9^i + c_2 3^i$$

$$T(3^i) = c_1 9^i + c_2 3^i$$

$$T(n) = c_1 (3^i)^2 + c_2 3^i$$

$$= c_1 n^2 + c_2 n$$

$$= \Theta(n^2)$$

Not always obvious what sort of transformation to do:

- Consider $T(n) = 2T(\sqrt{n}) + \log n$
- Let $n = 2^i$ and rewrite T(n):
- $T(2^i) = 2T(2^{i/2}) + i$
- Define $t(i) = T(2^i)$:
- t(i) = 2t(i/2) + i



Consider the recurrence T(n) = 2T(n/4) + kn, where T(1) = 1, and k is some constant

- Q1: What is the transformed recurrence t(i)? How do we rewrite n and T(n) to get this sequence?
- Q2: What is the annihilator of t(i)? What is the solution for the recurrence t(i)?
- Q3: What is the solution for T(n)? (i.e. do the reverse transformation)

- This final recurrence is something we know how to solve!
- $t(i) = O(i \log i)$
- The reverse transform gives:
 - $t(i) = O(i \log i) \tag{6}$
 - $T(2^i) = O(i \log i) \tag{7}$
 - $T(n) = O(\log n \log \log n)$ (8)

Todo _____

• HW 1

• Start Chapter 15 in text

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