CS 461, Lecture 4

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Today's Outline

- Annihilators for recurrences with non-homogeneous terms
- Transformations

Annihilator Method

- Write down the annihilator for the recurrence
- Factor the annihilator
- Look up the factored annihilator in the "Lookup Table" to get general solution
- Solve for constants of the general solution by using initial conditions

Lookup Table

$$(\mathbf{L} - a_0)^{b_0}(\mathbf{L} - a_1)^{b_1} \ldots (\mathbf{L} - a_k)^{b_k}$$

annihilates only sequences of the form:

$$\langle p_1(n)a_0^n + p_2(n)a_1^n + \ldots p_k(n)a_k^n \rangle$$

where $p_i(n)$ is a polynomial of degree $b_i - 1$ (and $a_i \neq a_j$, when $i \neq j$)

Examples

- Q: What does $(\mathbf{L}-3)(\mathbf{L}-2)(\mathbf{L}-1)$ annihilate?
- A: $c_0 1^n + c_1 2^n + c_2 3^n$
- Q: What does $(\mathbf{L}-3)^2(\mathbf{L}-2)(\mathbf{L}-1)$ annihilate?
- A: $c_0 1^n + c_1 2^n + (c_2 n + c_3) 3^n$
- Q: What does $(\mathbf{L}-1)^4$ annihilate?
- A: $(c_0 n^3 + c_1 n^2 + c_2 n + c_3) 1^n$
- Q: What does $(\mathbf{L}-1)^3(\mathbf{L}-2)^2$ annihilate?
- A: $(c_0 n^2 + c_1 n + c_2) 1^n + (c_3 n + c_4) 2^n$

Example

Consider the recurrence $T(n) = 7T(n-1) - 16T(n-2) + 12T(n-3)$, $T(0) = 1$, $T(1) = 5$, $T(2) = 17$

- **Write down the annihilator**: From the definition of the sequence, we can see that $\mathbf{L}^3 T - 7\mathbf{L}^2 T + 16\mathbf{L}T - 12T = 0$, so the annihilator is $\mathbf{L}^3 - 7\mathbf{L}^2 + 16\mathbf{L} - 12$
- **Factor the annihilator**: We can factor by hand or using a computer program to get $\mathbf{L}^3 - 7\mathbf{L}^2 + 16\mathbf{L} - 12 = (\mathbf{L}-2)^2(\mathbf{L}-3)$
- **Look up to get general solution**: The annihilator $(\mathbf{L}-2)^2(\mathbf{L}-3)$ annihilates sequences of the form $\langle (c_0 n + c_1) 2^n + c_2 3^n \rangle$
- **Solve for constants**: $T(0) = 1 = c_1 + c_2$, $T(1) = 5 = 2c_0 + 2c_1 + 3c_2$, $T(2) = 17 = 8c_0 + 4c_1 + 9c_2$. We've got three equations and three unknowns. Solving by hand, we get that $c_0 = 1, c_1 = 0, c_2 = 1$. **Thus:** $T(n) = n2^n + 3^n$

Example (II)

Consider the recurrence $T(n) = 2T(n-1) - T(n-2)$, $T(0) = 0$, $T(1) = 1$

- **Write down the annihilator**: From the definition of the sequence, we can see that $\mathbf{L}^2 T - 2\mathbf{L}T + T = 0$, so the annihilator is $\mathbf{L}^2 - 2\mathbf{L} + 1$
- **Factor the annihilator**: We can factor by hand or using the quadratic formula to get $\mathbf{L}^2 - 2\mathbf{L} + 1 = (\mathbf{L}-1)^2$
- **Look up to get general solution**: The annihilator $(\mathbf{L}-1)^2$ annihilates sequences of the form $(c_0 n + c_1) 1^n$
- **Solve for constants**: $T(0) = 0 = c_1$, $T(1) = 1 = c_0 + c_1$, We've got two equations and two unknowns. Solving by hand, we get that $c_0 = 0, c_1 = 1$. **Thus:** $T(n) = n$

At Home Exercise

Consider the recurrence $T(n) = 6T(n-1) - 9T(n-2)$, $T(0) = 1$, $T(1) = 6$

- Q1: What is the annihilator of this sequence?
- Q2: What is the factored version of the annihilator?
- Q3: What is the general solution for the recurrence?
- Q4: What are the constants in this general solution?

(Note: You can check that your general solution works for $T(2)$)

Non-homogeneous terms

- Consider a recurrence of the form $T(n) = T(n-1) + T(n-2) + k$ where k is some constant
- The terms in the equation involving T (i.e. $T(n-1)$ and $T(n-2)$) are called the *homogeneous* terms
- The other terms (i.e.k) are called the *non-homogeneous* terms

Example

- In a *height-balanced tree*, the height of two subtrees of any node differ by at most one
- Let $T(n)$ be the smallest number of nodes needed to obtain a height balanced binary tree of height n
- Q: What is a recurrence for $T(n)$?
- A: Divide this into smaller subproblems
  - To get a height-balanced tree of height n with the smallest number of nodes, need one subtree of height $n-1$, and one of height $n-2$, plus a root node
  - Thus $T(n) = T(n-1) + T(n-2) + 1$

Example

- Let's solve this recurrence: $T(n) = T(n-1) + T(n-2) + 1$ (Let $T_n = T(n)$, and $T = \langle T_n \rangle$)
- We know that $(\mathbf{L}^2 - \mathbf{L} - 1)$ annihilates the homogeneous terms
- Let's apply it to the entire equation:

$$\begin{aligned} (\mathbf{L}^2 - \mathbf{L} - 1)\langle T_n \rangle &= \mathbf{L}^2\langle T_n \rangle - \mathbf{L}\langle T_n \rangle - 1\langle T_n \rangle \\ &= \langle T_{n+2} \rangle - \langle T_{n+1} \rangle - \langle T_n \rangle \\ &= \langle T_{n+2} - T_{n+1} - T_n \rangle \\ &= \langle 1, 1, 1, \cdots \rangle \end{aligned}$$

Example

- This is close to what we want but we still need to annihilate $\langle 1, 1, 1, \cdots \rangle$
- It's easy to see that $\mathbf{L} - 1$ annihilates $\langle 1, 1, 1, \cdots \rangle$
- Thus $(\mathbf{L}^2 - \mathbf{L} - 1)(\mathbf{L} - 1)$ annihilates $T(n) = T(n-1) + T(n-2) + 1$
- When we factor, we get $(\mathbf{L} - \phi)(\mathbf{L} - \hat{\phi})(\mathbf{L} - 1)$, where $\phi = \frac{1+\sqrt{5}}{2}$ and $\hat{\phi} = \frac{1-\sqrt{5}}{2}$.

Lookup

- Looking up $(\mathbf{L}-\phi)(\mathbf{L}-\hat{\phi})(\mathbf{L}-1)$ in the table
- We get $T(n) = c_1\phi^n + c_2\hat{\phi}^n + c_3 1^n$
- If we plug in the appropriate initial conditions, we can solve for these three constants
- We'll need to get equations for $T(2)$ in addition to $T(0)$ and $T(1)$

General Rule

To find the annihilator for recurrences with non-homogeneous terms, do the following:

- Find the annihilator a_1 for the homogeneous part
- Find the annihilator a_2 for the non-homogeneous part
- The annihilator for the whole recurrence is then $a_1 a_2$

Another Example

- Consider $T(n) = T(n-1) + T(n-2) + 2$.
- The residue is $\langle 2, 2, 2, \cdots \rangle$ and
- The annihilator is still $(\mathbf{L}^2 - \mathbf{L} - 1)(\mathbf{L} - 1)$, but the equation for $T(2)$ changes!

Another Example

- Consider $T(n) = T(n-1) + T(n-2) + 2^n$.
- The residue is $\langle 1, 2, 4, 8, \cdots \rangle$ and
- The annihilator is now $(\mathbf{L}^2 - \mathbf{L} - 1)(\mathbf{L} - 2)$.

Another Example

- Consider $T(n) = T(n-1) + T(n-2) + n$.
- The residue is $\langle 1, 2, 3, 4, \cdots \rangle$
- The annihilator is now $(\mathbf{L}^2 - \mathbf{L} - 1)(\mathbf{L} - 1)^2$.

Another Example

- Consider $T(n) = T(n-1) + T(n-2) + n^2$.
- The residue is $\langle 1, 4, 9, 16, \cdots \rangle$ and
- The annihilator is $(\mathbf{L}^2 - \mathbf{L} - 1)(\mathbf{L} - 1)^3$.

Another Example

- Consider $T(n) = T(n-1) + T(n-2) + n^2 - 2^n$.
- The residue is $\langle 1-1, 4-4, 9-8, 16-16, \cdots \rangle$ and the
- The annihilator is $(\mathbf{L}^2 - \mathbf{L} - 1)(\mathbf{L} - 1)^3(\mathbf{L} - 2)$.

In Class Exercise

- Consider $T(n) = 3 * T(n-1) + 3^n$
- Q1: What is the homogeneous part, and what annihilates it?
- Q2: What is the non-homogeneous part, and what annihilates it?
- Q3: What is the annihilator of $T(n)$, and what is the general form of the recurrence?

Limitations

- Our method does not work on $T(n) = T(n-1) + \frac{1}{n}$ or $T(n) = T(n-1) + \lg n$
- The problem is that $\frac{1}{n}$ and $\lg n$ do not have annihilators.
- Our tool, as it stands, is limited.
- Key idea for strengthening it is *transformations*

Transformations Idea

- Consider the recurrence giving the run time of mergesort $T(n) = 2T(n/2) + kn$ (for some constant k), $T(1) = 1$
- How do we solve this?
- We have no technique for annihilating terms like $T(n/2)$
- However, we can *transform* the recurrence into one with which we can work

Transformation

- Let $n = 2^i$ and rewrite $T(n)$:
- $T(2^0) = 1$ and $T(2^i) = 2T(\frac{2^i}{2}) + k2^i = 2T(2^{i-1}) + k2^i$
- Now define a new sequence t as follows: $t(i) = T(2^i)$
- Then $t(0) = 1$, $t(i) = 2t(i-1) + k2^i$

Now Solve

- We've got a new recurrence: $t(0) = 1$, $t(i) = 2t(i-1) + k2^i$
- We can easily find the annihilator for this recurrence
- $(\mathbf{L}-2)$ annihilates the homogeneous part, $(\mathbf{L}-2)$ annihilates the non-homogeneous part, So $(\mathbf{L}-2)(\mathbf{L}-2)$ annihilates $t(i)$
- Thus $t(i) = (c_1 i + c_2)2^i$

Reverse Transformation

- We've got a solution for $t(i)$ and we want to transform this into a solution for $T(n)$
- Recall that $t(i) = T(2^i)$ and $2^i = n$

$$
\begin{aligned}
t(i) &= (c_1 i + c_2)2^i & (1)\\
T(2^i) &= (c_1 i + c_2)2^i & (2)\\
T(n) &= (c_1 \lg n + c_2)n & (3)\\
&= c_1 n \lg n + c_2 n & (4)\\
&= \Theta(n \lg n) & (5)
\end{aligned}
$$

Success!

Let's recap what just happened:

- We could not find the annihilator of $T(n)$ so:
- We did a *transformation* to a recurrence we could solve, $t(i)$ (we let $n = 2^i$ and $t(i) = T(2^i)$)
- We found the annihilator for $t(i)$, and solved the recurrence for $t(i)$
- We *reverse transformed* the solution for $t(i)$ back to a solution for $T(n)$

Another Example

- Consider the recurrence $T(n) = 9T(\frac{n}{3}) + kn$, where $T(1) = 1$ and k is some constant
- Let $n = 3^i$ and rewrite $T(n)$:
- $T(2^0) = 1$ and $T(3^i) = 9T(3^{i-1}) + k3^i$
- Now define a sequence t as follows $t(i) = T(3^i)$
- Then $t(0) = 1$, $t(i) = 9t(i-1) + k3^i$

Now Solve

- $t(0) = 1$, $t(i) = 9t(i-1) + k3^i$
- This is annihilated by $(\mathbf{L} - 9)(\mathbf{L} - 3)$
- So $t(i)$ is of the form $t(i) = c_1 9^i + c_2 3^i$

Reverse Transformation

- $t(i) = c_1 9^i + c_2 3^i$
- Recall: $t(i) = T(3^i)$ and $3^i = n$

$$\begin{aligned} t(i) &= c_1 9^i + c_2 3^i \\ T(3^i) &= c_1 9^i + c_2 3^i \\ T(n) &= c_1 (3^i)^2 + c_2 3^i \\ &= c_1 n^2 + c_2 n \\ &= \Theta(n^2) \end{aligned}$$

In Class Exercise

Consider the recurrence $T(n) = 2T(n/4) + kn$, where $T(1) = 1$, and k is some constant

- Q1: What is the transformed recurrence $t(i)$? How do we rewrite n and $T(n)$ to get this sequence?
- Q2: What is the annihilator of $t(i)$? What is the solution for the recurrence $t(i)$?
- Q3: What is the solution for $T(n)$? (i.e. do the reverse transformation)

A Final Example

Not always obvious what sort of transformation to do:

- Consider $T(n) = 2T(\sqrt{n}) + \log n$
- Let $n = 2^i$ and rewrite $T(n)$:
- $T(2^i) = 2T(2^{i/2}) + i$
- Define $t(i) = T(2^i)$:
- $t(i) = 2t(i/2) + i$

A Final Example

- This final recurrence is something we know how to solve!
- $t(i) = O(i \log i)$
- The reverse transform gives:

$$\begin{aligned} t(i) &= O(i \log i) && (6) \\ T(2^i) &= O(i \log i) && (7) \\ T(n) &= O(\log n \log \log n) && (8) \end{aligned}$$

Todo

- HW 1
- Start Chapter 15 in text