CS 530: Geometric and Probabilistic Methods in Computer Science
Final Exam (Fall '13)

1. Let $f(t) = e^{-\pi t^2}$, $f'(t) = -2\pi\, t\, e^{-\pi t^2}$, and $g(t) = at + b$. Prove or disprove the following: $\langle f', g \rangle = 0$ for all a and b.

2. Let $\mathcal{D}f = \frac{\partial f^2}{\partial t^2} + \frac{\partial f}{\partial t}$ where f is a function of t. The eigenfunctions of $\mathcal{D}$ are harmonic signals, $e^{j2\pi st}$, where s is frequency and t is time. Give an expression for the eigenvalue of $\mathcal{D}$ as a function of s.

3. The three vectors, $\mathbf{f}_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}^{\mathrm{T}}$, $\mathbf{f}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^{\mathrm{T}}$, $\mathbf{f}_3 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$, form a frame $\mathcal{F}$ for $\mathbb{R}^2$. Give a representation for the vector $\mathbf{v} = \begin{bmatrix} 1 & 2 \end{bmatrix}^{\mathrm{T}}$ in the frame. Show your work.

4. Let $\mathbf{x}_1 = \begin{bmatrix} 1 & -2 \end{bmatrix}^{\mathrm{T}}$, $\mathbf{x}_2 = \begin{bmatrix} -2 & 0 \end{bmatrix}^{\mathrm{T}}$, $\mathbf{x}_3 = \begin{bmatrix} 0 & -4 \end{bmatrix}^{\mathrm{T}}$, $\mathbf{x}_4 = \begin{bmatrix} -4 & 5 \end{bmatrix}^{\mathrm{T}}$, $\mathbf{x}_5 = \begin{bmatrix} 5 & 6 \end{bmatrix}^{\mathrm{T}}$, $\mathbf{x}_6 = \begin{bmatrix} 6 & 1 \end{bmatrix}^{\mathrm{T}}$, be samples of a two-dimensional vector random variable. Do the following:

   (a) Compute the mean of the random variable.

   (b) Compute the covariance matrix for the random variable (after the mean has been subtracted).

   (c) Compute the eigenvectors of the covariance matrix.

   (d) Compute the KL transform of $\begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$.

   (e) Compute the eigenvalues of the covariance matrix.

5. Consider the following stochastic matrix:

$$\mathbf{P} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{2} \end{bmatrix}.$$

(a) Draw the transition diagram for the Markov process.

(b) Is the Markov process irreducible? Aperiodic?

(c) Does a limiting distribution exist? If so, give it.

(d) What is the magnitude of **P**'s largest eigenvalue?

(e) Are the eigenvectors of **P** orthogonal?

(f) How many complex eigenvalues does **P** have?

6. Let $S_\ell(\lambda)$, $S_m(\lambda)$, and $S_s(\lambda)$ be the spectral sensitivity functions of the cones of the human retina. Let $D(\lambda)$ be the spectral reflectance function of a daffodil. Define a system of linear equations, which when solved, gives the relative amounts, $V_{700}(D)$, $V_{546}(D)$, and $V_{436}(D)$, of the three CIE standard primary sources, $\delta(\lambda - 700\text{ nm})$, $\delta(\lambda - 546\text{ nm})$, and $\delta(\lambda - 436\text{ nm})$, necessary to reproduce the color of a daffodil illuminated by white light.