RADM/PAPS Panel Discussion

CADE-18

William McCune
Mathematics and Computer Science Division
Argonne National Laboratory

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The Cancellative Semigroup Conjecture

GT: Group Theory

CS: Cancellative Semigroup

E_i: (universally quantified) equations in product and variables (no group identity, no inverse, no constants)

If

$$\text{GT}, E_1, \cdots, E_n \Longrightarrow E_m$$

then

$$\text{CS}, E_1, \cdots, E_n \Longrightarrow E_m$$

Support for the CS Conjecture

- $xyzyx = yxzxy \Longrightarrow uvvu = vuuv$
  - easy for GT (let z be the group identity)
  - not so easy for CS (hint: let z be wxy)
- $xxxyy = yxxyx \Longrightarrow uuvvv = vuvvu$
  - GT: invert both sides, invert each variable, simplify
  - CS: much more difficult

A Problem Related to the CS Conjecture

- GT definition of commutator: $[x, y] = x^{-1} \cdot y^{-1} \cdot x \cdot y$
- The following three statements are equivalent (Levi, 1942).

$$[[x, y], z] = [x, [y, z]] \qquad \text{(commutator is associative)}$$
$$[x \cdot y, z] = [x, z] \cdot [y, z] \qquad \text{(commutator distributes over product)}$$
$$[x, y] \cdot z = z \cdot [x, y] \qquad \text{(nilpotent class 2)}$$

- CS: we cannot define commutator, but we can state the property

$$y \cdot x \cdot [x, y] = x \cdot y$$

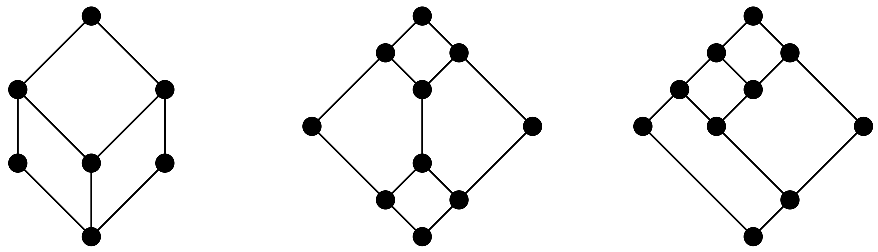
- Now, the three statements are equivalent in CS. (Otter, 2002).

Uniquely Complemented Lattices

- Lattice Theory (LT): an equational basis in $\wedge$ and $\vee$.
- Complementation: $x' \vee x = 1, \;\; x' \wedge x = 0$.
- Complements are unique: $x \vee y = 1, \;\; x \wedge y = 0 \;\;\Rightarrow\;\; x' = y$.
- *What makes UC lattices Boolean?* The so-called Huntington conditions.
  - finiteness
  - distributivity
  - modularity
  - *How weak can such a condition be?*

Uniquely Complemented Lattices

- Padmanabhan has a collection of 18 finite non-modular lattices, sizes 5–10. For example,



- He wanted to find Huntington equations true in as many as possible of the 18 lattices. That is, weak equations.

- Finding sets of candidates.

- Searching for proofs.