Problem 1.

(Exercise 2.42 in the text) Solve the recurrence $f(n) = \alpha f(n/2)$, with $f(1) = 1$. Since this recurrence is only defined for powers of 2, we write $n = 2^k$ and thus $f(2^k) = \alpha f(2^{k-1})$, or $g(k) = \alpha g(k-1)$, with $f(1) = g(0) = 1$. This yields the solution $g(k) = \alpha^k$ and thus $f(n) = \alpha^{\log_2 n} = n^{\log_\alpha 2}$.

Problem 2.

(Exercise 2.43 in the text) Solve the recurrence $f(n) = (f(n/2))^2$, with $f(1) = 1$. Observation immediately yields $f(n) = 1$—all that ever happens is that the previous value gets squared, but since we start with 1, nothing ever changes! If we have had $f(1) = c$, with $c > 1$, then we would have written $g(k) = (g(k-1))^2$ with $g(0) = c$ and used backsubstitution (this recurrence is not linear!) to obtain

$$g(k) = (g(k-1))^2 = ((g(k-2))^2)^2 = (g(k-2))^4 = \ldots = (g(0))^{2^k} = c^{2^k}$$

Reverting to n as a variable, we get $f(n) = c^{2^{\log_2 n}} = c^n$.

Problem 3.

(Exercise 2.44 in the text) Solve the recurrence $f(n) = (2 + \frac{1}{\log n}) f(n/2)$, with $f(1) = 1$. Since this is only defined for powers of two, we use our standard strategy to rewrite it as $g(k) = (2 + \frac{1}{k})g(k-1)$, with $g(0) = 1$. As it stands, this recurrence has non-constant coefficients, so we cannot solve it directly. We can easily bound it, however. Note that the additional term $\frac{1}{k}$ is always positive and never larger than 1. Thus, if we replace it by 0, we will get a lower bound; and if we replace it by 1, we will get an upper bound. Thus we get a lower bound function $lb(k) = 2lb(k-1)$, with $lb(0) = 1$, or $lb(k) = 2^k$; in terms of n, the lower bound is thus just n. The upper bound is defined by $ub(k) = 3ub(k-1)$, with $ub(0) = 1$, yielding $ub(k) = 3^k$; in terms of n, the upper bound is $n^{\log_2 3}$. Of the two bounds, the lower bound will be much tighter asymptotically, because $\frac{1}{k}$ gets infinitesimally small as k goes to infinity. Thus we can state that $f(n)$ is $O(n^{\log_2 3})$ and $\Omega(n)$.

All of which begs the question: is $f(n)$ in fact $\Theta(n)$? The answer is no, but that takes a bit more work. We would need to show that $f(n)$ does not grow faster than linearly. For that, we assume a specific form for f, then try verify that the right-hand side of the recurrence does not grow any faster than that specific form (which is the left-hand side). Thus write $f(n) = an + h(n)$, where $h(n)$ is $o(n)$, i.e., grows strictly more slowly than n. Then we would want to verify that we have

$$an + h(n) \geq (2 + \frac{1}{\log n}) f(n/2)$$

Simplifying terms, we get

$$h(n) \geq \frac{an}{2\log n/2} + (2 + \frac{1}{\log n})h(n/2)$$

or

$$h(n) \geq 2h(n/2) + \frac{1}{\log n} h(n/2) + \frac{an}{2 \log n}$$

which is only possible if $h(n)$ grows at least linearly (because of the first term on the right-hand side). But we have assumed that $h(n)$ grew strictly more slowly than linearly. Thus $f(n)$ is not $O(n)$. On the other hand, we can show, with just a bit more work, that $f(n)$ is $O(n^{1+\varepsilon})$ for any $\varepsilon > 0$—really all we need to observe is that $n^{1+\varepsilon}/\log n$ grows faster than linearly.

Problem 4.

Solve the following recurrences in Θ terms

1. $f(n) = 3f(n/2) + \Theta(n)$.
   This is only defined for n a power of 2, so we get the new recurrence $g(k) = 3g(k-1) + \Theta(2^k)$. This has a homogeneous root of 3, which dominates the driving term (root of 2), so its solution is $\Theta(3^k)$; thus $f(n)$ is $\Theta(n^{\log_2 3})$.

2. $f(n) = 4f(n-1) - 4f(n-2) + n^2 + 2^n$. The homogenous part has a double root of 2, so the homogeneous solution is of the form $(an+b)2^n$, which is $\Theta(n2^n)$. There are two driving terms, one a polynomial of degree 2 with an implicit root of 1 ($n^2 1^n$) and the other a polynomial of degree 0 with a root of 2. The first term gives rise to a similar term in the inhomogenous solution, that is, a $\Theta(n^2)$ term; the second term must have its degree increased by two to reflect the fact that there are two identical roots in the homogeneous part and so will give rise to a $\Theta(n^2 2^n)$ term in the inhomogeneous solution. Overall, that last term dominates everything, so that $f(n)$ is $\Theta(n^2 2^n)$.

3. $f(n) = 2f(n/2) - f(n/4) + \Theta(n)$.
   This is only defined for n a power of 2 (not a power of 4: if n is a power of 4, $n/2$ no longer is). Thus we get the new recurrence $g(k) = 2g(k-1) - g(k-2) + \Theta(2^k)$. The homogeneous part has a double root of 1 and so gives rise to a solution of the form $ak+b$, which is $\Theta(k)$; the driving term has no common root with the homogeneous root and so gives rise to an inhomogeneous solution of the same form, which is $\Theta(2^k)$. The latter dominates, so that $g(k)$ is $\Theta(2^k)$ and thus $f(n)$ is $\Theta(n)$.