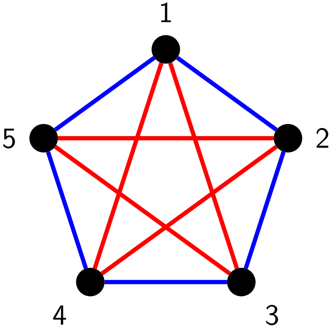
Amalgamations, Latin Squares, and Hamiltonian Decompositions

John Carr

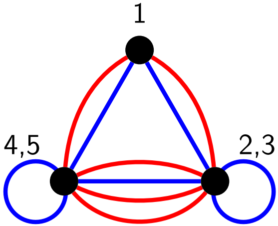
University of North Alabama

October 24, 2015

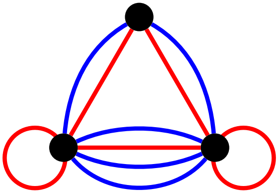
Amalgamation



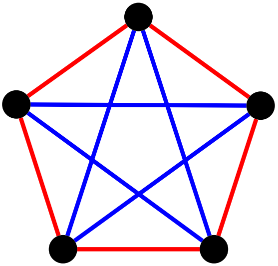
$\sim$



Amalgamation



$\overset{?}{\sim}$

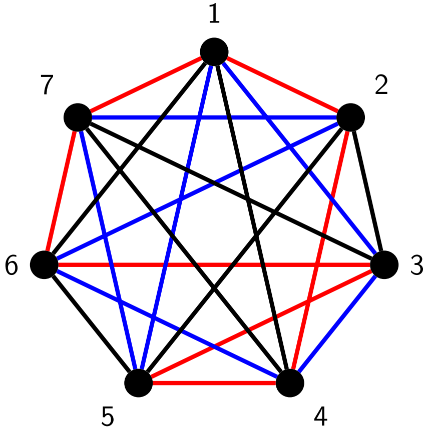


Definitions

- A graph is an ordered pair $G = (V, E)$ comprising a set V of vertices with a set E of edges.
- A complete graph (K_n) is a graph in which each pair of vertices is connected by an edge.
- An edge coloring is a labeling of the edges using colors. In particular, our coloring scheme allows for adjacent edges to have the same color.

Definitions

- A Hamiltonian path is a path in a graph that visits each vertex exactly once.
- A Hamiltonian cycle is a Hamiltonian path that is also a cycle (i.e. a path starting and ending with the same vertex).
- Determining whether such paths and cycles exist in graphs is the Hamiltonian path problem, which is NP-complete.
- A Hamiltonian Decomposition is a partition of the edge set of a graph into Hamiltonian cycles.



K_7 with 3 Hamiltonian Cycles

Latin Squares

a	b	c	d
b	c	d	a
c	d	a	b
d	a	b	c

A 4×4 Latin square

Definition

- A Latin square is an $n \times n$ array in which each cell contains a symbol from an alphabet of size n, such that each symbol in the alphabet occurs once and exactly once in each row and column.

Quasigroups

$(Q, \cdot)$	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

A Quasigroup of order 4

Definition

- A Quasigroup $(Q, \cdot)$, is an algebraic structure that can be represented by a Cayley table with the Latin square property.

Latin Squares and Amalgamation

1	3	2
3	2	1
2	1	3

A 3×3 Latin square

1	3	2	4
3	2	1	4
2	1	3	4
4	4	4	1

The original square amalgamated by 1

Note

The second square is no longer a Latin square. Is it possible to amalgamate the original Latin square to the point where the new square has no repeated elements in rows or columns?

Latin Squares and Amalgamation

1	3	2	4	5
3	2	1	5	4
2	1	3	4	5
4	5	4	3	2
5	4	5	2	3

The original square amalgamated by 2

1	3	2	4	5	6
3	2	1	5	6	4
2	1	3	6	4	5
4	5	6	1	2	3
5	6	4	2	3	1
6	4	5	3	1	2

The original square amalgamated by 3

Note

Amalgamating by 2 also doesn't work, but when we amalgamate by 3 we create a new Latin square of order 6.

Latin Squares and Amalgamation

1		2
	3	

An incomplete Latin square of order 3

1	4	2	
	3		

An incomplete Latin square of order 4

Note

We can also amalgamate to finish incomplete tables that are otherwise impossible to complete. See if you can complete the square!

Latin Squares and Amalgamation

1	4	2	3
2	3	4	1
3	2	1	4
4	1	3	2

A complete Latin square of order 4

Note

In this case, amalgamating by 1 allows us to complete the square. There are actually 8 ways to complete the square, and in total 576 unique Latin squares of order 4.

P-Quasigroups

Definition

- A P-Quasigroup $(Q, \cdot)$ is a quasigroup with the three following properties:
  1. $a \cdot a = a \; \forall a \in Q$ (Idempotence)
  2. $a \neq b \Rightarrow a \neq a \cdot b \neq b \; \forall a, b \in Q$
  3. $a \cdot b = c \iff c \cdot b = a \; \forall a, b, c \in Q$.

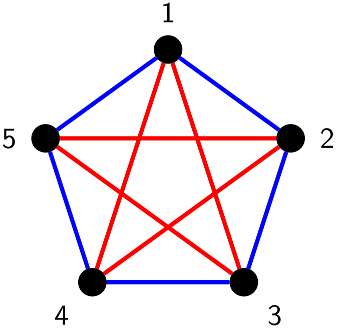
P-Quasigroups and Complete Graphs

Correspondence (Kotzig)

The correspondence between P-Quasigroups of n elements and decompositions of complete undirected graphs of n vertices into disjoint closed paths is established by labeling the vertices of the graph with the elements of Q and prescribing that the edges (a, b) and (b, c) shall belong to the same closed path if and only if $a \cdot b = c, a \neq b$.

Lemma (Kotzig)

The number of elements in a P-Quasigroup is odd.



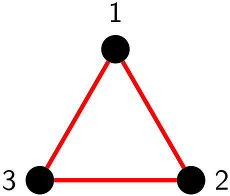
$\sim$

$(Q,\cdot)$	1	2	3	4	5
1	1	3	5	2	4
2	5	2	4	1	3
3	4	1	3	5	2
4	3	5	2	4	1
5	2	4	1	3	5

Note

The same decomposition of the complete graph used in the beginning actually corresponds to a P-Quasigroup of order 5.

P-Quasigroups and Hamiltonian Decompositions



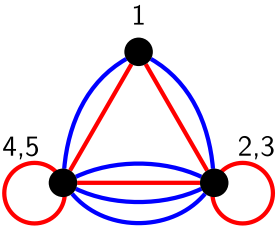
$\sim$

$(Q, \circ)$	1	2	3
1	1	3	2
2	3	2	1
3	2	1	3

Question

Recall that in the beginning we amalgamated K_5 down to a version of K_3 with extra loops and edges. What happens if we try to amalgamate this P-Quasigroup of order 3 up to a P-Quasigroup of order 5?

P-Quasigroups and Hamiltonian Decompositions



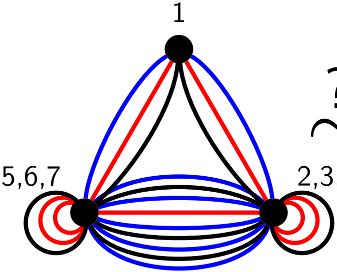
$\overset{?}{\sim}$

$(Q,\circ)$	1	2	3	4	5
1	1	3	2	5	4
2	3	2	1	5	4
3	2	1	3	5	4
4	5	5	5	4	2
5	4	4	4	2	5

Note

We run into the same issue we saw earlier when we amalgamated Latin squares! We know we can amalgamate to size 6 in order to solve the issue, but P-Quasigroups have odd order. What about amalgamating to size 7?

P-Quasigroups and Hamiltonian Decompositions



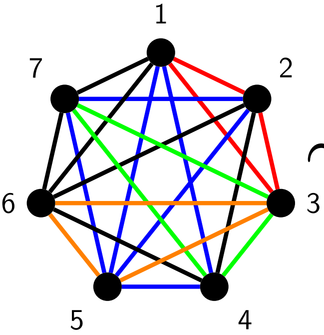
$\overset{?}{\sim}$

$(Q,\circ)$	1	2	3	4	5	6	7
1	1	3	2	5	7	4	6
2	3	2	1	6	4	7	5
3	2	1	3	7	6	5	4
4	5	6	7	4	2	1	3
5	4	7	6	1	5	3	2
6	7	4	5	2	3	6	1
7	6	5	4	3	1	2	7

Note

We now have a P-Quasigroup of order 7, but does the graph still contain Hamiltonian cycles?

P-Quasigroups and Hamiltonian Decompositions



$\sim$

$(Q,\circ)$	1	2	3	4	5	6	7
1	1	3	2	5	7	4	6
2	3	2	1	6	4	7	5
3	2	1	3	7	6	5	4
4	5	6	7	4	2	1	3
5	4	7	6	1	5	3	2
6	7	4	5	2	3	6	1
7	6	5	4	3	1	2	7

Note

After disentangling the previous graph, we see this graph doesn't contain any Hamiltonian cycles. What causes this?

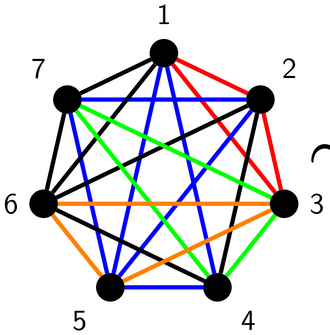
P-Quasigroups and Hamiltonian Decompositions

Theorem

If a P-Quasigroup $(Q, \cdot)$ of order n corresponds to a Hamiltonian Decomposition (HD) of a complete graph (G) of order n, then Q doesn't contain a subquasigroup.

Proof.

For the sake of contradiction, assume Q corresponds to a HD in graph G and that Q contains a proper subquasigroup F. Since F is closed under the same operation of Q, we know $\exists x \in Q$ such that $x \notin F$. Then $\forall\ a, b \in F$ we have that $ab \neq x$ and we have a closed path that doesn't touch every vertex. Therefore Q doesn't correspond to a HD, a contradiction. $\square$



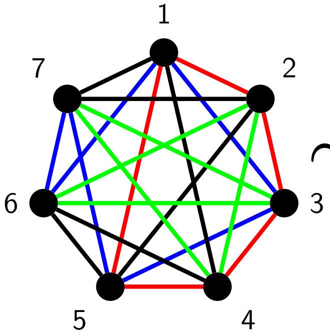
$\sim$

$(Q,\circ)$	1	2	3	4	5	6	7
1	1	3	2	5	7	4	6
2	3	2	1	6	4	7	5
3	2	1	3	7	6	5	4
4	5	6	7	4	2	1	3
5	4	7	6	1	5	3	2
6	7	4	5	2	3	6	1
7	6	5	4	3	1	2	7

Note

Q has a subquasigroup $F = \{1, 2, 3\}$.

P-Quasigroups and Hamiltonian Decompositions



$\sim$

$(Q,\circ)$	1	2	3	4	5	6	7
1	1	3	5	6	4	7	2
2	5	2	4	7	6	3	1
3	6	1	3	5	7	2	4
4	7	6	2	4	1	5	3
5	2	7	1	3	5	4	6
6	3	4	7	1	2	6	5
7	4	5	6	2	3	1	7

Note

This P-Quasigroup doesn't contain a subquasigroup but the graph is not partitioned into Hamiltonian Cycles.

Current Work

- What are all of the necessary and sufficient conditions such that a P-Quasigroup corresponds to a Hamiltonian decomposition?
- Using P-Quasigroups, what is a general way of disentangling an amalgamated graph that preserves the structure of the original graph?

Thanks!

- Kotzig, "Groupoids and Partitions of Complete Graphs." *Combinatorial Structures and their Applications*, University of Calgary, Alta., Canada (1969)