CS 361, Lecture 12

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Outline

- Review
- Limitations of current methods
- Domain and Range Transformations
- Recap of Annihilators

Lookup Table

$$(\mathbf{L} - a_0)^{b_1}(\mathbf{L} - a_1)^{b_2} \dots (\mathbf{L} - a_k)^{b_k}$$

annihilates only sequences of the form:

$$\langle p_1(n)a_0^n + p_2(n)a_1^n + \dots p_k(n)a_k^n \rangle$$

where $p_i(n)$ is a polynomial of degree $b_i - 1$ (and $a_i \neq a_j$, when $i \neq j$)

Non-homogeneous terms

- Consider a recurrence of the form $T(n) = T(n-1) + T(n-2) + k$ where k is some constant
- The terms in the equation involving T (i.e. $T(n-1)$ and $T(n-2)$) are called the *homogeneous* terms
- The other terms (i.e.k) are called the *non-homogeneous* terms

General Rule

To find the annihilator for recurrences with non-homogeneous terms, do the following:

- Find the annihilator a_1 for the homogeneous part
- Find the annihilator a_2 for the non-homogeneous part
- The annihilator for the whole recurrence is then $a_1 a_2$

Example

- In a *height-balanced tree*, the height of two subtrees of any node differ by at most one
- Let $T(n)$ be the smallest number of nodes needed to obtain a height balanced binary tree of height n
- Q: What is a recurrence for $T(n)$?
- A: Divide this into smaller subproblems
  - To get a height-balanced tree of height n, need one subtree of height $n-1$, and one of height $n-2$, plus a root node
  - Thus $T(n) = T(n-1) + T(n-2) + 1$

Example

- Consider recurrence $T(n) = T(n-1) + T(n-2) + 1$
- Homogeneous part: $T_1(n) = T(n-1) + T(n-2)$
- Non-homogeneous part: $T_2(n) = 1$
- $(\mathbf{L}^2 - \mathbf{L} - 1)$ annihilates the homogeneous part (T_1)
- $(\mathbf{L} - 1)$ annihilates the non-homogeneous part (T_2)
- So $(\mathbf{L}^2 - \mathbf{L} - 1)(\mathbf{L} - 1)$ annihilates the sequence

Example (II)

- Factoring gets $(\mathbf{L} - \phi)(\mathbf{L} - \hat{\phi})(\mathbf{L} - 1)$, where $\phi = \frac{1+\sqrt{5}}{2}$ and $\hat{\phi} = \frac{1-\sqrt{5}}{2}$.
- Look up gives us that: $T(n) = c_1\phi^n + c_2\hat{\phi}^n + c_3 1^n$

Another Example

- Consider $T(n) = T(n-1) + T(n-2) + 2^n$.
- Homogeneous part: $T_1(n) = T(n-1) + T(n-2)$
- Non-homogeneous part: $T_2(n) = 2^n$
- $(\mathbf{L}^2 - \mathbf{L} - 1)$ annihilates the $T_1(n) = T(n-1) + T(n-2)$
- $(L - 2)$ annihilates $T_2(n) = 2^n$
- So $(\mathbf{L}^2 - \mathbf{L} - 1)(L - 2)$ is the annihilator of $T(n)$

In Class Exercise

- Consider $T(n) = 3 * T(n-1) + 3^n$
- Q1: What is the homogeneous part, and what annihilates it?
- Q2: What is the non-homogeneous part, and what annihilates it?
- Q3: What is the annihilator of $T(n)$, and what is the general form of the recurrence?

Limitations

- Our method does not work on $T(n) = 2T(n/2) + n$, $T(n) = T(n/4) + 1$ or $T(n) = T(n-1) + \lg n$
- The problem is that $2T(n/2)$, $\frac{1}{n}$ and $\lg n$ do not have annihilators.
- Our tool, as it stands, is limited
- Key idea for strengthening it is *transformations*

Transformations Idea

- Consider the recurrence giving the run time of mergesort $T(n) = 2T(n/2) + kn$ (for some constant k), $T(1) = 1$
- How do we solve this?
- We have no technique for annihilating terms like $T(n/2)$
- However, we can *transform* the recurrence into one with which we can work

Transformation

- Let $n = 2^i$ and rewrite $T(n)$:
- $T(2^0) = 1$ and $T(2^i) = 2T(\frac{2^i}{2}) + k2^i = 2T(2^{i-1}) + k2^i$
- Now define a new sequence t as follows: $t(i) = T(2^i)$
- Then $t(0) = 1$, $t(i) = 2t(i-1) + k2^i$

Now Solve

- We've got a new recurrence: $t(0) = 1$, $t(i) = 2t(i-1) + k2^i$
- We can easily find the annihilator for this recurrence
- $(\mathbf{L}-2)$ annihilates the homogeneous part, $(\mathbf{L}-2)$ annihilates the non-homogeneous part, So $(\mathbf{L}-2)(\mathbf{L}-2)$ annihilates $t(i)$
- Thus $t(i) = (c_1 i + c_2)2^i$

Reverse Transformation

- We've got a solution for $t(i)$ and we want to transform this into a solution for $T(n)$
- Recall that $t(i) = T(2^i)$ and $2^i = n$

$$\begin{aligned} t(i) &= (c_1 i + c_2)2^i & (1)\\ T(2^i) &= (c_1 i + c_2)2^i & (2)\\ T(n) &= (c_1 \lg n + c_2)n & (3)\\ &= c_1 n \lg n + c_2 n & (4)\\ &= \Theta(n \lg n) & (5) \end{aligned}$$

Success!

Let's recap what just happened:

- We could not find the annihilator of $T(n)$ so:
- We did a *transformation* to a recurrence we could solve, $t(i)$ (we let $n = 2^i$ and $t(i) = T(2^i)$)
- We found the annihilator for $t(i)$, and solved the recurrence for $t(i)$
- We *reverse transformed* the solution for $t(i)$ back to a solution for $T(n)$

Another Example

- Consider the recurrence $T(n) = 9T(\frac{n}{3}) + kn$, where $T(1) = 1$ and k is some constant
- Let $n = 3^i$ and rewrite $T(n)$:
- $T(2^0) = 1$ and $T(3^i) = 9T(3^{i-1}) + k3^i$
- Now define a sequence t as follows $t(i) = T(3^i)$
- Then $t(0) = 1$, $t(i) = 9t(i-1) + k3^i$

Now Solve

- $t(0) = 1$, $t(i) = 9t(i-1) + k3^i$
- This is annihilated by $(\mathbf{L}-9)(\mathbf{L}-3)$
- So $t(i)$ is of the form $t(i) = c_1 9^i + c_2 3^i$

Reverse Transformation

- $t(i) = c_1 9^i + c_2 3^i$
- Recall: $t(i) = T(3^i)$ and $3^i = n$

$$\begin{aligned} t(i) &= c_1 9^i + c_2 3^i \\ T(3^i) &= c_1 9^i + c_2 3^i \\ T(n) &= c_1 (3^i)^2 + c_2 3^i \\ &= c_1 n^2 + c_2 n \\ &= \Theta(n^2) \end{aligned}$$

In Class Exercise

Consider the recurrence $T(n) = 2T(n/4) + kn$, where $T(1) = 1$, and k is some constant

- Q1: What is the transformed recurrence $t(i)$? How do we rewrite n and $T(n)$ to get this sequence?
- Q2: What is the annihilator of $t(i)$? What is the solution for the recurrence $t(i)$?
- Q3: What is the solution for $T(n)$? (i.e. do the reverse transformation)

A Final Example

Not always obvious what sort of transformation to do:

- Consider $T(n) = 2T(\sqrt{n}) + \log n$
- Let $n = 2^i$ and rewrite $T(n)$:
- $T(2^i) = 2T(2^{i/2}) + i$
- Define $t(i) = T(2^i)$:
- $t(i) = 2t(i/2) + i$

A Final Example

- This final recurrence is something we know how to solve!
- $t(i) = n \log i$
- The reverse transform gives:

$$t(i) = i \log i \quad (6)$$

$$T(2^i) = i \log i \quad (7)$$

$$T(n) = \log n \log \log n \quad (8)$$