CS 361, Lecture 26

Jared Saia
University of New Mexico

Outline

- Skip Lists

Project

- Any questions on the group project? (hw6)

Administrative

- Lab Section evaluation this week
- This week, Kanglin will take attendance at sections, if you're there, you'll get an extra check for participation
- Sections are Thursday 3:30-4:20 and Friday 1:00-1:50
- Good chance to review material for final

Project

- Project will be due May 8th in class
- Late projects will *not* be accepted
- You can get partial credit for an unfinished project turned in on time but will get no credit for a finished project turned in late

HW

- There will also be a hw due on May 8th in class
- This will be a "final review" hw

Final

- Final will be Tuesday May 13th, 7:30-9:30am in our regular classroom
- Closed book, but two pieces of paper are allowed (for cheat sheets)
- No calculators

High Level Analysis

Comparison of various BSTs

- RB-Trees: + guarantee $O(\log n)$ time for each operation, easy to augment, − high constants
- AVL-Trees: + guarantee $O(\log n)$ time for each operation, − high constants
- B-Trees: + works well for trees that won't fit in memory, guarantee $O(\log n)$ time for each operation, − inserts and deletes are more complicated
- Splay Tress: + small constants, − amortized guarantees only
- Skip Lists: + easy to implement, − runtime guarantees are probabilistic only

Which Data Structure to use?

- Splay trees work very well in practice, the "hidden constants" are small
- Unfortunately, they can not guarantee that *every* operation takes $O(\log n)$
- When this guarantee is required, B-Trees are best when the entire tree will not be stored in memory
- If the entire tree will be stored in memory, RB-Trees, AVL-Trees, and Skip Lists are good

Skip List

- Technically, not a BST, but they implement all of the same operations
- Very elegant randomized data structure, simple to code but analysis is subtle
- They guarantee that, with high probability, all the major operations take $O(\log n)$ time

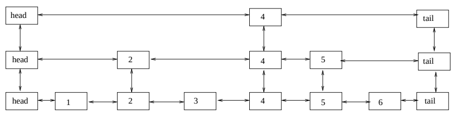
Skip List

- A skip list is basically a collection of doubly-linked lists, $L_1, L_2, \ldots, L_x$, for some integer x
- Each list has a special head and tail node, the keys of these nodes are assumed to be −MAXNUM and +MAXNUM respectively
- The keys in each list are in sorted order (non-decreasing)

Skip List

- Every key is in the list L_1.
- For all $i > 2$, if a key k is in the list L_i, it is also in L_{i-1}. Further there are up and down pointers between the k in L_i and the k in L_{i-1}.
- All the head(tail) nodes from neighboring lists are interconnected

Example



Search

```
Search(k){
  pLeft = L_x.head;
  for (i=x;i>=0;i--){
    Search from pLeft in L_i to get the rightmost elem, r,
      with value <= k;
    pLeft = pointer to r in L_(i-1);
  }
  if (pLeft==k)
    return pLeft
  else
    return nil
  }
}
```


Insert

p is a constant between 0 and 1, typically $p = 1/2$, let rand() return a random value between 0 and 1

```
Insert(k){
First call Search(k), let pLeft be the leftmost elem <= k in L_1
Insert k in L_1, to the right of pLeft
i = 2;
while (rand()<= p){
  insert k in the appropriate place in L_i;
}
```


Deletion

- Deletion is very simple
- First do a search for the key to be deleted
- Then delete that key from all the lists it appears in from the bottom up, making sure to "zip up" the lists after the deletion

In-Class Exercise Trick

A trick for computing expectations of discrete positive random variables:

- Let X be a discrete r.v., that takes on values from 1 to n

$$E(X) = \sum_{i=1}^{n} P(X \geq i)$$

Why?

$$\begin{aligned}
\sum_{i=1}^{n} P(X \geq i) &= P(X=1) + P(X=2) + P(X=3) + \ldots \\
&+ P(X=2) + P(X=3) + P(X=4) + \ldots \\
&+ P(X=3) + P(X=4) + P(X=5) + \ldots \\
&+ \ldots \\
&= 1 * P(X=1) + 2 * P(X=2) + 3 * P(X=3) + \ldots \\
&= E(X)
\end{aligned}$$

In-Class Exercise

Q: How much memory do we expect a skip list to use up?

- Let X_i be the number of lists that element i is inserted in.
- Q: What is $P(X_i \geq 1)$, $P(X_i \geq 2)$, $P(X_i \geq 3)$?
- Q: What is $P(X_i \geq k)$ for general k?
- Q: What is $E(X_i)$?
- Q: Let $X = \sum_{i=1}^{n} X_i$. What is $E(X)$?

Height of Skip List

- Assume there are n nodes in the list
- Q: What is the probability that a particular key i achieves height exceeding $k \log n$ for some constant k?
- A: If $p = 1/2$, $P(X_i \geq k \log n) = \frac{1}{n^k}$

Height of Skip List

- Q: What is the probability that *any* of the nodes achieve height higher than $k \log n$?
- A: We want

$$P(X_1 \geq k \log n \text{ or } X_2 \geq k \log n \text{ or } \dots \text{ or } X_n \geq k \log n)$$

- By a Union Bound, this probability is no more than

$$P(X_1 \geq k \log n) + P(X_2 \geq k \log n) + \cdots + P(X_n \geq k \log n)$$

- Which equals $\frac{n}{n^k} = n^{1-k}$

Height of Skip List

- If we choose k to be, say 10, this probability gets very small as n gets large
- In particular, the probability of having a skip list of size exceeding $k \log n$ is $o(1)$
- So we say that the height of the skip list is $O(\log n)$ *with high probability*

Search Time

- Note that the expected number of "siblings" of a node, x, at any level i is 2
- Why? Because for a node to be a sibling of x at level i, it must have failed to advance to the next level
- The first node that advances to the next level ends the possibility of further siblings.
- This is the same as asking expected number of times we need to flip a coin to get a heads.

Flipping to get Heads

- How many times in expectation do we need to flip a coin to get heads, if the coin is heads with probability p?
- Let X be a random variable giving the number of times the coin is flipped until we get heads, then $E(X)$ is the expected number of times needed to flip to get heads
- Then $E(X) = 1 + (1-p)E(X)$ since we take 1 flip, plus in the case where the coin is tails (which happens with probability $(1-p)$), we then take "the expected number of times needed to flip to get heads" (i.e. we're no better off than when we started)
- Solving for $E(X)$ gives $E(X) = 1/p$. If $p = 1/2$, then $E(X) = 2$

Search Time

- The expected number of “siblings” of a node, x, at any level i is 2
- The number of levels is $O(\log n)$ with high probability
- From these two facts, we can prove that the expected search time is $O(\log n)$ (the proof is omitted)
- (Warning: The argument is not as simple as multiplying these two values. We can't do this since the two random variables are not independent.)