CS 361, Lecture 8

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Outline

“Partly because of his computational skills, Gerbert, in his later years, was made Pope by Otto the Great, Holy Roman Emperor, and took the name Sylvester II. By this time, his gift in the art of calculating contributed to the belief, commonly held throughout Europe, that he had sold his soul to the devil.”
- Dominic Olivastro in the book Ancient Puzzles, *1993*

- Loop Invariants
- Binary Heaps

Correctness of Algorithms

- The most important aspect of algorithms is their correctness
- An algorithm by definition *always* gives the right answer to the problem
- A procedure which doesn’t always give the right answer is a *heuristic*
- All things being equal, we prefer an algorithm to a heuristic
- How do we prove an algorithm is really correct?

Loop Invariants

A useful tool for proving correctness is loop invariants. Three things must be shown about a loop invariant

- **Initialization:** Invariant is true before first iteration of loop
- **Maintenance:** If invariant is true before iteration i, it is also true before iteration $i+1$ (for any i)
- **Termination:** When the loop terminates, the invariant gives a property which can be used to show the algorithm is correct

Example Loop Invariant

- We'll prove the correctness of a simple algorithm which solves the following interview question:
- *Find the middle of a linked list, while only going through the list once*
- The basic idea is to keep two pointers into the list, one of the pointers moves twice as fast as the other
- (Call the head of the list the 0-th elem, and the tail of the list the $(n-1)$-st element, assume that $n-1$ is an even number)

Example Algorithm

```
GetMiddle (List l){
  pSlow = pFast = l;
  while ((pFast->next)&&(pFast->next->next)){
    pFast = pFast->next->next
    pSlow = pSlow->next
  }
  return pSlow
}
```


Example Loop Invariant

- *Invariant: At the start of the i-th iteration of the while loop, pSlow points to the i-th element in the list and pFast points to the $2i$-th element*
- **Initialization:** True when $i = 0$ since both pointers are at the head
- **Maintenance:** if pSlow, pFast are at positions i and $2i$ respectively before i-th iteration, they will be at positions $i+1$, $2(i+1)$ respectively before the $i+1$-st iteration
- **Termination:** When the loop terminates, pFast is at element $n-1$. Then by the loop invariant, pSlow is at element $(n-1)/2$. Thus pSlow points to the middle of the list

Challenge

- Figure out how to use a similar idea to determine if there is a loop in a linked list *without marking nodes!*

What is a Heap

- "A heap data structure is an array that can be viewed as a nearly complete binary tree"
- Each element of the array corresponds to a value stored at some node of the tree
- The tree is completely filled at all levels except for possibly the last which is filled from left to right

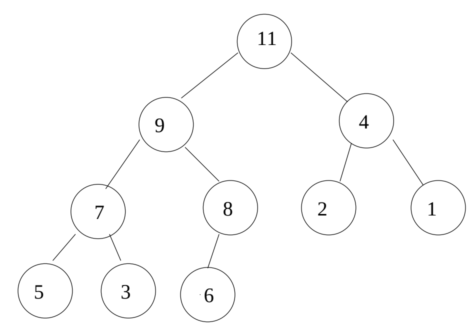
heap-size (A)

- An array A that represents a heap has two attributes
  - length (A) which is the number of elements in the array
  - heap-size (A) which is the number of elems in the heap stored within the array
- I.e. only the elements in A[1..heap-size (A)] are elements of the heap

Tree Structure

- A[1] is the root of the tree
- For all i, $1 < i <$ heap-size (A)
  - Parent (i) $= \lfloor i/2 \rfloor$
  - Left (i) $= 2i$
  - Right (i) $= 2i + 1$
- If Left (i) $>$ heap-size (A), there is no left child of i
- If Right (i) $>$ heap-size (A), there is no right child of i
- If Parent (i) < 0, there is no parent of i

Example



A:

1	2	3	4	5	6	7	8	9	10
11	9	4	7	8	2	1	5	3	6

Max-Heap Property

- For every node i other than the root, A[Parent (i)] $\geq$ A[i]

Max-Heap Property

- For every node i other than the root, A[Parent (i)] $\geq$ A[i]
- Parent is always at least as large as its children
- Largest element is at the root

(A Min-heap is organized the opposite way)

Height of Heap

- Height of a node in a heap is the number of edges in the longest simple downward path from the node to a leaf
- Height of a heap of n elements is $\Theta(\log n)$. Why?

Maintaining Heaps

- Q: How to maintain the heap property?
- A: *Max-Heapify* is given an array and an index i. Assumes that the binary trees rooted at $Left(i)$ and $Right(i)$ are max-heaps, but $A[i]$ may be smaller than its children.
- *Max-Heapify* ensures that after its call, the subtree rooted at i is a Max-Heap

Max-Heapify

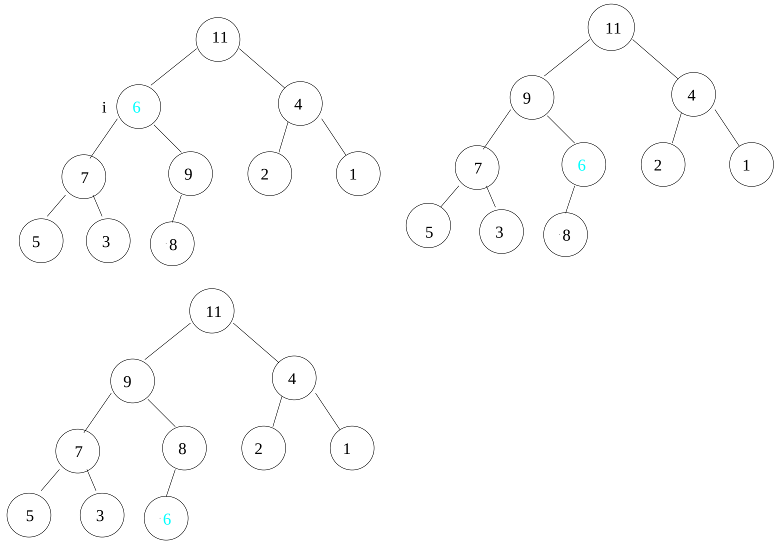
- Main idea of the Max-Heapify algorithm is that it percolates down the element that start at $A[i]$ to the point where the subtree rooted at i is a max-heap
- To do this, it repeatedly swaps $A[i]$ with its largest child until $A[i]$ is bigger than both its children
- For simplicity, the algorithm is described recursively.

Max-Heapify

Max-Heapify (A,i)

1. $l = Left(i)$
2. $r = Right(i)$
3. $largest = i$
4. if ($l \leq$ heap-size(A) and $A[l] > A[i]$) then $largest = l$
5. if ($r \leq$ heap-size(A) and $A[r] > A[largest]$) then $largest = r$
6. if $largest \neq i$ then
   (a) exchange A[i] and A[largest]
   (b) Max-Heapify (A,largest)

Example



Analysis

- Let $T(h)$ be the runtime of max-heapify on a subtree of height h
- Then $T(1) = \Theta(1)$, $T(h) = T(h-1) + 1$
- Solution to this recurrence is $T(h) = \Theta(h)$
- Thus if we let $T(n)$ be the runtime of max-heapify on a subtree of *size* n, $T(n) = O(\log n)$, since $\log n$ is the maximum height of heap of size n

Build-Max-Heap

- Q: How can we convert an arbitrary array into a max-heap?
- A: Use Max-Heapify in a bottom-up manner
- Note: The elements A[$\lfloor n/2 \rfloor + 1$],..,A[n] are all leaf nodes of the tree, so each is a 1 element heap to begin with

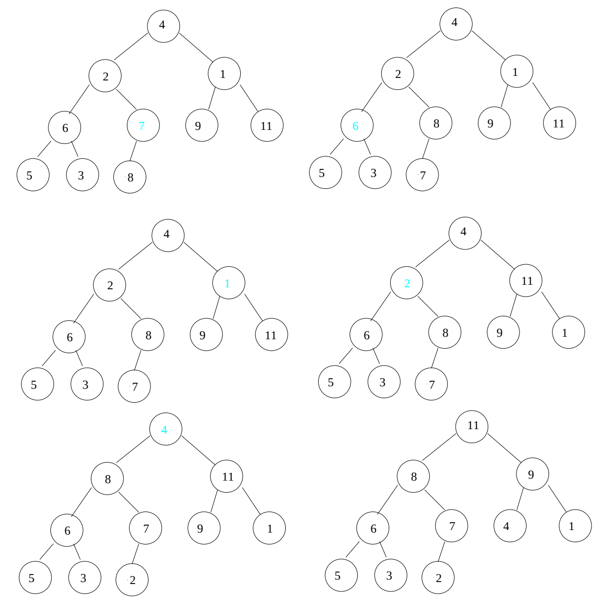
Build-Max-Heap

Build-Max-Heap (A)

1. heap-size (A) = length (A)
2. for ($i = \lfloor length(A)/2 \rfloor$;$i > 0$;$i - -$)
   (a) do Max-Heapify (A,i)

Example

A = 4 2 1 6 7 9 11 5 3 8



Loop Invariant

- Loop Invariant: "At the start of the i-th iteration of the for loop, each node $i+1, i+2, \ldots n$ is the root of a max-heap"

Correctness

- **Initialization:** $i = \lfloor n/2 \rfloor$ prior to first iteration. But each node $\lfloor n/2 \rfloor + 1$, $\lfloor n/2 \rfloor + 2, \ldots, n$ is a leaf so is the root of a trivial max-heap
- **Termination:** At termination, $i = 0$, so each node $1, \ldots, n$ is the root of a max-heap. In particular, node 1 is the root of a max heap.

Maintenance

- **Maintenance:** First note that if the nodes $i+1, \ldots n$ are the roots of max-heaps before the call to Max-Heapify (A,i), then they will be the roots of max-heaps after the call. Further note that the children of node i are numbered higher than i and thus by the loop invariant are both roots of max heaps. Thus after the call to Max-Heapify (A,i), the node i is the root of a max-heap. Hence, when we decrement i in the for loop, the loop invariant is established.

Time Analysis

(Naive) Analysis:

- Max-Heapify takes $O(\log n)$ time per call
- There are $O(n)$ calls to Max-Heapify
- Thus, the running time is $O(n \log n)$

Time Analysis

Better Analysis. Note that:

- An n element heap has height no more than $\log n$
- There are at most $n/2^{h+1}$ nodes of any height h (to see this, consider the min number of nodes in a heap of height h)
- Time required by Max-Heapify when called on a node of height h is $O(h)$.
- Thus total time is: $\sum_{h=0}^{\log n} \frac{n}{2^{h+1}} O(h)$

Analysis

$$\sum_{h=0}^{\log n} \frac{n}{2^{h+1}} O(h) = O\left(n \sum_{h=0}^{\log n} \frac{h}{2^h}\right) \quad (1)$$

$$= O\left(n \sum_{h=0}^{\infty} \frac{h}{2^h}\right) \quad (2)$$

$$= O(n) \quad (3)$$

Analysis

The last step follows since for all $|x| < 1$,

$$\sum_{i=0}^{\infty} i x^i = \frac{x}{(1-x)^2} \quad (4)$$

Can get this equality by recalling that for all $|x| < 1$,

$$\sum_{i=0}^{\infty} x^i = \frac{1}{1-x},$$

and taking the derivative of both sides!

Heap-Sort

Heap-Sort (A)

1. Build-Max-Heap (A)
2. for (i=length (A);$i > 1$;$i - -$)
   (a) do exchange A[1] and A[i]
   (b) heap-size (A) = heap-size (A) - 1
   (c) Max-Heapify (A,1)

Analysis

- Build-Max-Heap takes $O(n)$, and each of the $O(n)$ calls to Max-Heapify take $O(\log n)$, so Heap-Sort takes $O(n \log n)$
- Correctness???