Spring 2005 Theory Comprehensive Exam
Answer all 6 questions.

1. EXAM SEATING is the following problem: I have a set S of students who are taking a final exam, and a symmetric function $\text{Cheat} : S \times S \to \{0, 1\}$ such that $\text{Cheat}(s_1, s_2) = \text{Cheat}(s_2, s_1) = 1$ if s_1 and s_2 will copy from each others' exams if they can and 0 if they won't. I am trying to design an order for the students to sit in a line; each student can see the exam of the students to their left and right. Question: is there an ordering of the students in a line so that no cheating can take place?

   Prove that EXAM SEATING is NP-complete. Hint: think of Cheat as describing a graph.

2. Suppose you are flying from Albuquerque, NM to Bangkok, Thailand, and you are given the prices of flights between any two cities in the world.

   (a) How do you schedule your trip if you want to spend the least amount of money on airplane tickets?

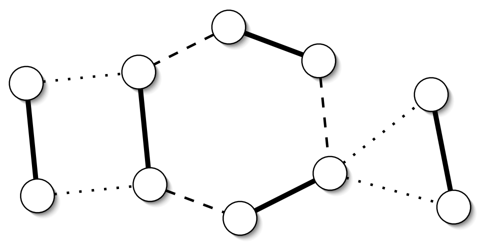
   (b) Now suppose you don't want to have too many stops on your trip. How do you schedule your trip if you want to have at most k stops while also spending as little money as possible? Is this problem polynomial-time solvable, as a function of n (the number of cities) and/or k (the number of stops)? Why or why not?

   (c) What if you want to visit exactly k cities during the trip while still wanting to spend as little money as possible? Is the problem still polynomial-time solvable? Why or why not?

3. Assume you are given a weighted undirected connected graph $G = (V, E)$, where all the edge weights are unique, and some edge $e \in E$. Give an $O(|E|)$-time algorithm to determine if e is in the minimum-weight spanning tree for G. Prove that your algorithm is correct.

4. Suppose I store n items in a hash table of size n according to a good pseudorandom hash function, so that each item is hashed to a uniformly random location. Prove that with high probability, no location in the table receives $\log n$ or more items.

5. Given a graph $G = (V, E)$, a *perfect matching* is a subset of $M \subseteq E$ such that each vertex appears in exactly one $m \in M$. (In particular, $|M| = |V|/2$.) Given a graph G, let G' be the (much larger!) graph whose vertices are the perfect matchings of G, and where an edge exists between two perfect matchings if you can get from one to the other by the following kind of "flip" move: 1) find an even cycle C of G such that every vertex in C is matched with an adjacent vertex in C, and then 2) replace the edges of C in M with those in $C \setminus M$. For instance, in the figure below, the current matching consists of the bold edges. The hexagon forms such a cycle C, and we can replace the solid edges with the dashed edges:



and we could do the same with the square on the left.

Prove that G' is connected for any G; in other words, we can get from any perfect matching to any other through a sequence of moves of this type.

6. The theory of NP-completeness addresses the worst-case complexity of problems. However, most real-world instances are not designed by an adversary. Does this mean the theory of NP-completeness has little practical value? Write a *short* comment (two paragraphs maximum) on the extent to which NP-completeness is or is not relevant to the real world, and what other ideas in computer science can help us understand and deal with real-world instances of NP-complete problems.