Spring 2007 Theory Comprehensive Exam

Answer any 5 out of 9 questions. State clearly which questions you want us to grade; we will only grade those 5, and only they will contribute to your score. The exam is closed-book and closed-notes.

If you run out of time, the idea of the prood is much more important (and will earn more partial credit) than mathematical formalism.

1. Suppose I have a graph $G = (V, E)$. Recall that an *independent set* is a subset $S \subset V$ which does not contain any neighboring pairs of vertices.

   An *interval graph* is an undirected graph where each vertex v corresponds to an interval $[a, b]$ on the real line, and where two vertices are connected if the corresponding intervals intersect.

   Given the set of intervals corresponding to the vertices of an interval graph G, design an efficient algorithm to find the maximum independent set in G. For instance, given the set of intervals $\{[1, 2], [1.6, 2.7], [2.5, 4]\}$, there is an independent set $\{[1, 2], [2.5, 4]\}$ of size 2.

   We all know, of course, that finding the maximum independent set problem is an NP-hard problem. Discuss the implications, if any, of your algorithm given this fact.

2. Suppose I have a weighted graph $G = (V, E)$, where each edge e has a positive weight $w(e)$. I want to construct a *maximum-weight spanning tree*, i.e., a spanning tree T which maximizes the total weight of the edges in T. Design a polynomial-time algorithm for this problem and prove that it works.

3. Given an integer sequence $a_1, a_2, \ldots, a_n$, an *increasing subsequence of length k* is a subset $a_{i_1} < a_{i_2} < \cdots < a_{i_k}$ with $i_1 < i_2 < \cdots < i_k$. For example, the sequence

   $$6, 2, 17, 4, 9, 8, 10, 5, 12$$

   has an increasing subsequence of length 5:

   $$2, 4, 8, 10, 12$$

   Describe an algorithm that, given a sequence of length n, finds its longest possible subsequence in time polynomial in n.

4. A graph $G = (V, E)$ is *3-colorable* if it is possible to color the vertices red, blue and green so that no two vertices of the same color are connected. Suppose you have access to a friendly oracle which answers the yes-or-no question of whether a graph is 3-colorable. Show that if G is 3-colorable, then you can find a 3-coloring in polynomial time by asking this oracle a polynomial number of questions.

5. Give an example of a function which is computable, but which is *provably* not in P. Explain the proof (at least the idea of it).

6. Rectangle Puzzle is the following problem. Given a large rectangle whose dimensions are $X \times Y$, and a set of n smaller rectangles of dimension $x_1 \times y_1, x_2 \times y_2, \ldots, x_n \times y_n$ and whose total area is $\sum_i x_i y_i = XY$, is it possible to arrange the small rectangles so that they cover the large one with no gaps or overlaps? Prove that this problem is NP-complete. Assume that the dimensions X, Y and the x_i, y_i have a polynomial number of digits.

7. Hazel and Naomi each have a brown paper grocery bag containing three exotic fruits. Initially, Hazel's bag contains three persimmons and Naomi's bag contains three kumquats. Once every day, Hazel and Naomi swap one fruit chosen at random from their bags.

   (a) Derive expression for the probabilities $P(k+1 \mid k)$, $P(k \mid k)$ and $P(k-1 \mid k)$ that Hazel will have $k+1$, k, or $k-1$ kumquats today given that she had k kumquats yesterday.

   (b) Give the transition matrix for the Markov process.

   (c) Is this Markov process irreducible? aperiodic? Prove your answers.

8. A bivariate Gaussian random variable, $\mathbf{x} = \begin{bmatrix} x_0 & x_1 \end{bmatrix}^{\mathrm{T}}$, has the following p.d.f.:

$$f(x_0, x_1) = \frac{\sqrt{a^2 - b^2}}{2\pi} \exp\left(-\frac{1}{2} \left[a x_0^2 + 2b x_0 x_1 + a x_1^2 \right] \right).$$

   (a) Let $\mathbf{u} = \mathbf{W}\mathbf{x}$. Give an expression for $g(u_0, u_1)$, the p.d.f. for the bivariate Gaussian random variable, $\mathbf{u} = \begin{bmatrix} u_0 & u_1 \end{bmatrix}^{\mathrm{T}}$.

   (b) Give the matrix $\mathbf{W}$ which will decorrelate the components of $\mathbf{x}$.