Perturbation Methods with Nonlinear Changes of Variables

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April, 2002

Abstract. The perturbation method is commonly used to approximate solutions to dynamic economic models. Two types of Taylor series are conventionally used: Taylor series in x and Taylor series in $\ln x$. We combine the perturbation method with a continuum of changes of variables and show that substantially better expansions can be achieved at little extra cost.

Economists are studying increasingly complex dynamic stochastic models and need more powerful and reliable computational methods. Most previous work has focussed on computing linear approximations of equilibrium relations. Some recent work (Judd and Guu (1993, 1997), Gaspar and Judd (1997), Judd (1998), Collard and Juillard (2001), and Jin and Judd (2001)) have extended this approach by computing Taylor series approximations of equilibrium relations and produce increasingly accurate approximations through increasing the order of the Taylor series. However, computational costs increase substantially as the order increases. This paper shows how to use nonlinear changes of variables to improve the accuracy of perturbation method solutions at low computational cost.

Linearization methods for dynamic models have been a workhorse of macroeconomic analysis. Magill (1977) showed how to compute a linear approximation around deterministic steady states and apply them to approximate spectral properties of stochastic models. Kydland and Prescott (1982) applied a special case of the Magill method to a real business cycle model. However, the approximations in Magill, and Kydland and Prescott were just linear approximations of the deterministic model applied to stochastic models; they ignored higher-order terms and were certainty equivalent approximations, that is, variance had no impact on decision rules. The motivating intuition was also specific to the case of linear, certainty equivalent approximations. Kydland and Prescott (1982) motivated their procedure by replacing the nonlinear law of motion with an equivalent linear law of motion[1], replacing the nonlinear payoff function with a quadratic approximation, and then applying linear-quadratic dynamic programming methods to the approximate model. This motivation gives the impression that it is not easy to compute higher-order approximations, particularly since computing the first-order terms requires solving a quadratic matrix equation. In fact, Marcet(1994) dismissed the possibility that higher-order approximations be computed, stating that "perturbation methods of order higher than one are considerably more complicated than the traditional linear-quadratic case ..."

Furthermore, little effort has been made to determine the conditions under which certainty equivalent linearizations are valid. Linearization methods are typically used in an application without examining whether they are valid in that case. This raises questions about many of the applications, particularly since the conventional linearization approach sometimes produces clearly erroneous results. For example, Tesar (1995) uses the standard Kydland-Prescott method and found an example where completing asset markets will make all agents worse off. This result violates general equilibrium theory and can only be attributed to the numerical method used. Kim and Kim (forthcoming) show that this will often occur in simple stochastic models. Jin and Judd (2001) presents a portfolio-like example where casual applications of higher-order procedures (such as those advocated by Sims, 2002, and Campbell and Viciera, 2002) can easily produce answers inconsistent with the model they try to approximate. These examples emphasize two important points. First, more flexible, robust, and accurate methods based on sound mathematical principles are needed. Second, we cannot blindly accept the results of a Taylor series approximation but need ways to test an approximation's reliability.

More recent work have used the implicit function theorem and Taylor series methods to go beyond the normal "linearize around the steady state" approximations by adding both higher-order terms and deviations from certainty equivalence. This work has shown that higher-order approximations are straightforward to do, even, in some sense, *easier* to compute then the linear term. Judd and Guu (1993, 1997)

[1]They used a change of variables, not a linear approximation, to create the substitute problem.

used Mathematica to examine perturbation methods for deterministic, continuous- and discrete-time growth models in one capital stock, and stochastic growth models in continuous time with one state. They find that the high-order approximations can be used to compute highly accurate approximations which avoid the certainty equivalence property of the standard linearization method. Judd and Gaspar (1997) described perturbation methods for multidimensional stochastic models in continuous time, and produced Fortran computer code for fourth-order expansions. Judd (1998) presented the general method for deterministic discrete-time models and presented a discrete-time stochastic example indicating the critical adjustments necessary to move from continuous time to discrete time. In particular, the natural perturbation parameter is the instantaneous variance in the continuous-time case, but the standard deviation is the natural perturbation parameter for discrete-time stochastic models. Jin and Judd (2001) extended these methods to more general rational expectations models, focussing on the structure of the problem, existence issues, and error evaluation procedures. The reader is referred to these papers and their mathematical sources for key definitions and introductions to these methods. Higher-order methods have been applied to a variety of particular problems. Collard and Juillard (2001) computed a higher-order perturbation approximation of an asset-pricing model. Kim and Kim (forthcoming) applied second-order approximation methods to welfare questions in international trade. Sims (2000) and Grohe-Schmidt and Uribe (2002) have generalized Judd (1998), Judd and Gaspar (1997), and Judd and Guu (1993) by examining second-order approximations of multidimensional discrete-time models.

These analyses have a common structure. Suppose that x is the state, or predetermined, variable, and u is a free variable. Equilibrium takes the form of a feedback rule $u = U(x)$. Together with the law of motion $x_{t+1} = f(x_t, u_t)$, this leads to an equilibrium law of motion $x_{t+1} = f(x_t, U(x_t))$. The linear approximation is $U(x) = u_* + U'(x_*)(x - x_*)$, and higher-order approximations are similar Taylor series. Sometimes economists execute this approach in terms of other variables. For example, one can express equilibrium in terms of $\ln x$ and/or $\ln u$. This will create linear approximations of the form $u = a \ln x + b$, $\ln u = ax + b$, or the log-log form $\ln u = a \ln x + b$.

There have been many reasons given for expressing equilibrium in one of these alternatives. For example, a linear approximation to a consumption policy $C(k)$ may occasionally imply a negative consumption but a log-log approximation will guarantee a positive consumption. Campbell (1994) argues that a log-linear approximation "gives a simpler relation between the parameters of the underlying model and the parameters that appear in the approximate solution." But some of these reasons are not important. For example, if the shocks to technology are so great that a linear approximation produces a negative consumption then the linear approximation is probably a poor one even when it produces positive consumption. Also, one can use the linear approximations to produce simple relations between the solutions and the model parameters (see, for example, Judd, 1987). This paper takes the view that accuracy is the only appropriate objective in choosing among alternative Taylor series expansions, and that conventional criterion are, at best, derivative of this accuracy objective.

This paper first argues that these alternatives are all just a few examples of a broad family of alternatives defined by nonlinear changes of variables. This point is rather obvious but is largely ignored in the literature. The second point is that one can convert a Taylor series in one form to a Taylor series in any other form at little cost since the coefficients for one series are linearly related to the coefficients of the ordinary Taylor series expansion. Third, one can also check the quality of any of these polynomial approximations at little cost. Therefore, this paper studies the following two-stage strategy. First, we first compute an ordinary Taylor series approximation. This step can be expensive since it involves computing high-order derivatives. Second, we compute alternative power series implied by several possible changes of variables and examine their quality in terms of its implied Euler equation errors. This step can be comparatively cheap.

We apply this strategy to a simple growth model. We find that one can quickly find alternative expansions which dominate the usual ones by two orders of magnitude. The problem of finding the absolute best nonlinear change of variable is a difficult problem since the objective is multimodal. However, we find that it is easy to find something that dominates the conventional choices. We show that the multidimensional case is a straightforward extension of the one-dimensional case.

The purpose of this paper is to introduce the reader to the basic ideas and explore its potential value.

1. TAYLOR SERIES APPROXIMATIONS AND NONLINEAR CHANGES OF VARIABLES

We first describe how to combine Taylor series approximations with changes of variables (COV) to produce asymptotically valid expressions in an arbitrary variable. This will give us a general view of the problem and help us understand how standard procedures fit into the more general approach described here.

1.1. Taylor Series Expansions and COVs. The Taylor series expansion of $f(x)$ at $x = a$ is

$$f(a) + (x-a)\,f'(a) + \frac{(x-a)^2\,f''(a)}{2} + \frac{(x-a)^3\,f^{(3)}(a)}{6} + \ldots$$

Define the nonlinear change of variables

$$y = Y(x)$$

where $Y(x)$ is strictly increasing and nonlinear in x. The new variable y is related nonlinearly to the initial variable x. We assume that Y is a nonlinear function; if $Y(x)$ were linear in x then the resulting series would be identical to the series in x. Also define the corresponding inverse function

$$x = X(y)$$

We can express the function $f(x)$ in terms of the new variable y by defining the function $g(y)$

$$g(y) = f(X(y))$$

If we knew $g(y)$ then we could use it to compute f since $f(x) = g(Y(x))$. We will not know g explicitly but we can use a Taylor series approximation of $g(y)$ based at $y = b = Y(a)$ to construct an approximation of $f(x)$ near $x = 1$. The Taylor series for g is constructed from its derivatives at $y = b$, which are derived by using the Chain Rule. The result is

$$\begin{aligned}
g\left(y\right) &= f\left(X\left(y\right)\right) \doteq f(X(b)) + (y-b)\ f'(X(b))\,X'(b) + \\
&\frac{1}{2}\left(y-b\right)^2\left(X'(b)^2\,f''(X(b)) + f'(X(b))\,X''(b)\right) \\
&+\frac{1}{6}\left(y-b\right)^3\left(X'(b)\,f''(X(b))\,X''(b) + 3X'(b)^3\,f^{(3)}(X(b)) + f'(X(b))\,X^{(3)}(b)\right) \\
&= f(a) + (y-b)\ f'(a)X'(b) \\
&+\frac{1}{2}\left(y-b\right)^2\left(X'(b)^2\,f''(a) + f'(a)\,X''(b)\right) \\
&+\frac{1}{6}\left(y-b\right)^3\left(X'(b)\,f''(a)\,X''(b) + 3X'(b)^3\,f^{(3)}(a) + f'(a)\,X^{(3)}(b)\right)
\end{aligned}$$

The Taylor series for g is constructed from derivatives of f and X, both of which are assumed to be known. Also, these derivatives are computed in a direct manner. In fact, the derivatives are linear in the derivatives of f. Since $f\left(x\right) = g\left(Y\left(x\right)\right)$ and $b = Y\left(a\right)$ we have the following $Y\left(x\right)$-expansion for $f\left(x\right)$

$$\begin{aligned}
f\left(x\right) &= g\left(Y\left(x\right)\right) \doteq f(a) + (Y(x)-b)\ f'(a)\,X'(b) \\
&+\frac{1}{2}\left(Y(x)-b\right)^2\ \left(X'(b)^2\,f''(a) + f'(a)\,X''(b)\right) \\
&+\frac{1}{6}\left(Y(x)-b\right)^3\ \left(X'(b)\,f''(a)\,X''(b) + 2X'(b)^3\,f^{(3)}(a) + f'(a)\,X^{(3)}(b)\right)
\end{aligned}$$

Again we see that the $Y\left(x\right)$-expansion is computed directly from the ordinary derivatives of f, and, moreover, the coefficients are linear functions of those derivatives. Therefore, constructing the $Y\left(x\right)$-expansion of f is a trivial computational task once we know the ordinary Taylor series expansion of f at $x = a$.

More generally, we will want to transform both the the domain and range of $f\left(x\right)$. For example, economists often express a consumption policy $C\left(k\right)$ by expressing $\ln C\left(k\right)$ as a linear function of k. Specifically, this approach expresses some nonlinear function of $f\left(x\right)$ as a polynomial in some variable $Y\left(x\right)$. This is formalized by finding some function h such that

$$h\left(f\left(x\right)\right) = g\left(Y\left(x\right)\right)$$

This implies

$$h\left(f\left(X\left(y\right)\right)\right) = g\left(y\right)$$

The derivatives of $g(y)$ are given by the Chain Rule, and equal

$$\begin{aligned} g'(y) &= h'(f(X(y)))\, f'(X(y))\, X'(y) \\ g''(y) &= h''(f(X(y)))\left(f'(X(y))\, X'(y)\right)^2 \\ &\quad + h'(f(X(y)))\, f''(X(y))\, X'(y)^2 \\ &\quad + h'(f(X(y)))\, f'(X(y))\, X''(y) \end{aligned}$$

the Taylor series of $g(y)$ is

$$\begin{aligned} g^{Tay}(Y(x)) &= h(f(a)) + (Y(x)-b)\, g'(b) + \frac{1}{2}(Y(x)-b)^2\, g''(b) + ... \\ g'(b) &= h'(f(a))\, f'(a)\, X'(b) \\ g''(b) &= h''(f(a))\left(f'(a)\, X'(b)\right)^2 \\ &\quad + h'(f(a))\, f''(a)\, X'(b)^2 \\ &\quad + h'(f(a))\, f'(a)\, X''(b) \end{aligned}$$

and the approximation of $f(x)$ becomes

$$\begin{aligned} f(x) &= h^{-1}(g(Y(x))) \doteq h^{-1}\left(g^{Tay}(Y(x))\right) \\ &= h^{-1}\left(h(f(a)) + (Y(x)-b)\, g'(b) + \frac{1}{2}(Y(x)-b)^2\, g''(b) + ...\right) \end{aligned}$$

These formulas indicate the general idea. We now turn to some familiar and some unfamiliar applications of these ideas.

1.2. Examples of COVs. Suppose we want to approximate a function $f(x)$ near $x = x_0$. The normal Taylor series in levels is

$$f(x) = f(x_0) + (x - x_0)\, f'(x_0) + \frac{(x-x_0)^2\, f''(x_0)}{2} + ...$$

We next display polynomial approximations corresponding to some familiar and some unfamiliar COVs.

Log-Log Expansion. A common example of this method is the construction of log-log approximations. In our notation above, this implies $Y(x) = \ln x$ and

$h(z) = \ln z$. We can also use the derivatives of f at $x = x_0$ to create the log-log approximation. If we define the first- and second-order elasticities to be

$$\eta_1 = \frac{x_0 f'(x_0)}{f(x_0)},\ \eta_2 = \frac{x_0^2 f''(x_0)}{f(x_0)}$$

then the second-order log-log approximation of f in terms of $\ln x$ can be expressed as

$$\ln f(x) - \ln f(x_0) \doteq \eta_1(\ln x - \ln x_0) + (\eta_1(1-\eta_1) + \eta_2)\frac{(\ln x - \ln x_0)^2}{2}$$

The first portion of this approximation is the familar log-linear approximation. The quadratic term is not so familiar but is a straightforward extension of the conventional log-linear approximation of f.

Power Function Expansions. More generally, we can consider using the power function in our the changes of variables. This implies the choices

$$\begin{aligned} Y(x) &= x^\alpha,\ X(y) = y^{1/\alpha} \\ h(x) &= x^\gamma,\ h^{-1}(x) = x^{1/\gamma} \end{aligned}$$

Notice that we use different power functions for Y and h. We shall call this COV the $\alpha - \gamma$ power function COV. This change of variables implies the first-order expansion

$$f(x) \doteq \left(f(x_0)^\gamma + (x^\alpha - x_0^\alpha)\frac{\gamma x_0^{1-\alpha} f(x_0)^{-1+\gamma} f'(x_0)}{\alpha} \right)^{1/\gamma}$$

if neither α nor γ is zero. In terms of the elasticity η_1 this reduces to

$$f(x) \doteq f(x_0)\left(1 - \eta_1\left(1 - \left(\frac{x}{x_0}\right)^\alpha\right)\frac{\gamma}{\alpha}\right)^{1/\gamma}$$

The power-power expansions are generalizations of both the log-log and the ordinary expansions. The choices $\alpha = \gamma = 1$ produce the ordinary Taylor series expansion. In a limiting sense, the log COV is the case of $\alpha = 0$ ($\gamma = 0$). Therefore, when one is implementing the power function COV, the case of $\alpha = 0$ is a special case which is the log COV.

Other possibilities. We next display some other possibilities to illustrate the large range of alternatives available to a modeller. We do not systematically explore these possibilities in our example below, but we present them to emphasize the point that we only need a function to be monotone increasing and easily invertible.

One simple possible choice is the canonical rational function,

$$y = \frac{a + bx}{c + dx}$$

with inverse

$$x = \frac{a - cy}{dy - b}$$

This change of variable is well-defined and monotone over any interval over which $c + dx$ is never zero. Without loss of generality, one could normalize this COV by restricting the coefficients so that $a + b = c + d = 1$, implying the family

$$y = \frac{a + (1 - a)\,x}{c + (1 - c)\,x}$$

with inverse

$$x = \frac{a - cy}{a - 1 + (1 - c)\,y}$$

Without loss of generality, two COV's related by an affine transformation are equivalent. Therefore, the canonical rational COV is really just the one-parameter family

$$y = \frac{x}{c + (1 - c)\,x}$$

Another possibility is the quadratic rational function

$$y = \frac{x + ax^2}{b + cx + (1 - b - c)x^2}$$

which is monotone over some intervals and has an explicit inverse. Our final example is

$$y = \left(\frac{x^\alpha + ax^{2\alpha}}{b + cx^\alpha + (1 - b - c)x^{2\alpha}}\right)^\gamma$$

which has five free parameters and combines the quadratic rational function family with the power-power family.

2. A Simple Growth Model Example

We now illustrate how to use COVs in a simple perturbation problem. We use the simple optimal growth problem

$$\begin{array}{ll} \max_{c_t} & \sum_{t=0}^{\infty} \beta^t \, u(c_t), \\ \text{s.t.} & k_{t+1} = F(k_t) - c_t. \end{array} \tag{1}$$

The solution can be expressed as a policy function, $C(k)$, satisfying the Euler equation

$$u'\left(C(k)\right) = \beta \, u'\left(C\left(F(k) - C(k)\right)\right) \, F'\left(F(k) - C(k)\right). \tag{2}$$

At the steady state, k^*, we have $F(k^*) - C(k^*) = k^*$, where (2) implies that $u'\left(C(k^*)\right) = \beta \, u'\left(C(k^*)\right) \, F'(k^*)$, which in turn implies the steady state condition $1 = \beta F'(k^*)$ which uniquely determines k^*. Furthermore $k^* = F(k^*) - C(k^*)$ determines $C(k^*)$.

This is a good example since it will allow us to focus on the critical issues. In particular, it nicely illustrates the method. There are different ways to express an approximation to $C(k)$. Some will just compute an approximation of C which is linear in k but others argue that it is better to express $\ln C$ as a linear function of $\ln k$. In fact, there is no general correct choice. If u is quadratic and F is linear then the true solution for $C(k)$ is linear in k, but if F is Cobb-Douglas and u is the log function then the true solution for C expresses $\ln C$ as a linear function of $\ln k$. These examples are well-known and appear to argue for one of these two choices. However, if u has nearly infinite curvature, then the solution for $C(k)$ is nearly equal to $F(k) - k$, which is net output, and will then take on whatever functional form $F(k) - k$ has. These examples illustrate the various functional forms the solution may take. However, in general, the true solution does not fall into any of these categories. Therefore, we should examine several alternative COVs before settling on a solution.

We will examine the case where

$$\begin{aligned} u(c) &= \ln c \\ \beta &= .95 \\ F(k) &= k + \frac{4}{19} k^{1/4} \end{aligned}$$

which implies a steady state capital stock of $k^* = 1$.

2.1. Ordinary Perturbation. We first describe the method by which we compute the ordinary Taylor series approximation. Taking the derivative of (2) with respect to k implies

$$\begin{aligned} u''(C)\,C' &= \beta u''(C(F-C))\,C'(F-C)\,\left[F'-C'\right]F'(F-C) \\ &+\beta\, u'(C(F(k-C)))\;\; F''(F-C)\,\left[F'-C'\right]. \end{aligned} \tag{3}$$

At $k=k^*$, (3) reduces to (we will now drop all arguments)

$$u''C' = u''C'\left[F'-C'\right]+\beta u'\,F''\left[F'-C'\right]. \tag{4}$$

In (4) we know the value of all the terms at $k=k^*$ except $C'(k^*)$. Equation (4) is a quadratic equation in $C'(k^*)$ with the solution

$$C' = \frac{1}{2}\left(1-F'+\beta\frac{u'}{u''}\;\;F''+\sqrt{\left(-1+F'-\beta\,\frac{u'}{u''}\,F''\right)^2+4\,\frac{u'}{u''}\,F''}\right). \tag{5}$$

We next compute higher-order terms of the Taylor expansion of $C(k)$ at $k=k^*$. If we take another derivative of (3) and set $k=k^*$, we find that $C''(k^*)$ must satisfy

$$\begin{aligned} u''C'' &+u'''C'C' \\ &= \beta u'''\left(C'F'(1-C')\right)^2\;F'+\beta u''C''\left(F'(1-C')\right)^2\;F' \\ &+2\beta u''C'F'(1-C')^2\,F''+\beta u'F'''(1-C')^2+\beta u'F''(-C''). \end{aligned} \tag{6}$$

The key fact is that (6) is is a linear equation in the unknown $C''(k^*)$. This analysis can continue to compute higher-order terms; see Judd and Guu(1993) for Mathematica programs to solve this problem. We continue this up to order 12 in our example.

2.2. Error Evaluation. We will use a stringent error criterion to judge the approximations arising from various COVs. We first define the normalized Euler equation error function. Suppose that $\hat{C}(k)$ is our approximation; we then define

$$E(k) = \left|1-\beta\frac{u'\left(\hat{C}\left(F(k)-\hat{C}(k)\right)\right)\;F'\left(F(k)-\hat{C}(k)\right)}{u'\left(\hat{C}(k)\right)}\right|. \tag{7}$$

which is the Euler equation error at k relative to current marginal utility $u'\left(\hat{C}(k)\right)$. This expression is unit-free and is therefore an appropriate index of accuracy. More specifically, if $E(k) < \varepsilon$ for all k then $\hat{C}(k)$ is an ε-equilibrium in the sense that the equilibrium conditions are satisfied up to an error less than ε.

We will use $E(k)$ in two ways. First, we display graphs of $E(k)$ to see how the apparent accuracy of $\hat{C}(k)$ changes as we change k. Second, we will compute scalar indices of error

$$E^{\infty}(a,b) = \max_{k\in[a,b]} \log_{10} E(k)$$

which is the maximum error over the interval $[a,b]$ expressed in terms of its base 10 logarithm. If $E^{\infty}(a,b)$ is less that -5, for example, then that means that the Euler equations are at most 10^{-5} for $k \in [a,b]$.

2.3. Error Evaluation of the Ordinary Perturbation Methods. Figure 1 displays the Euler equation errors at various capital stocks k and for various orders of approximation using the standard Taylor series. We see that the approximations decay as k moves away from the steady state $k = 1$, but that at each k the approximation is uniformly better as we increase the order of the approximation.

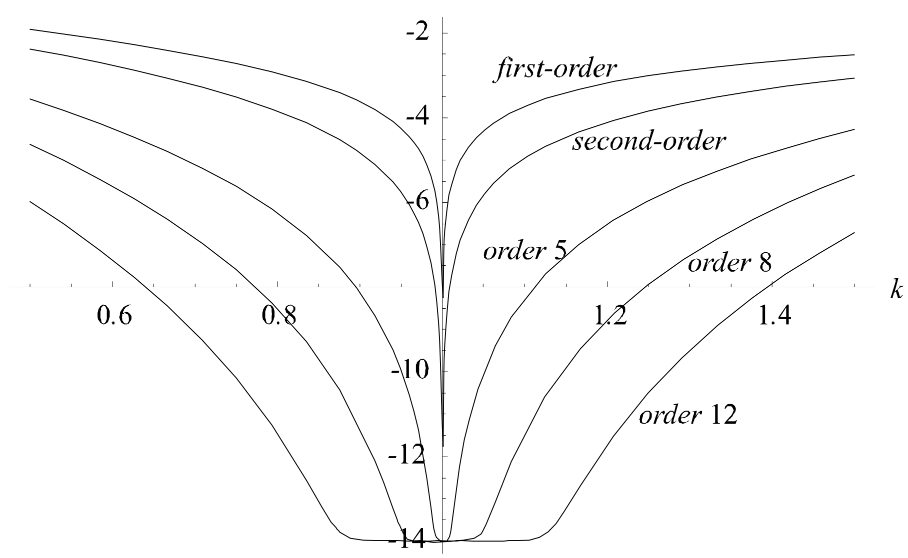

Figure 1: Errors with ordinary polynomial Taylor series

2.4. Errors Using the Log-Log Change of Variables. Figure 2 displays the Euler equation errors at various capital stocks k and for various orders of the log-log approximation. We again see that the approximations decay as k moves away from the steady state $k = 1$, but that at each k the approximation is uniformly better as we increase the order of the approximation.

We can also compare the log-log approximations with the ordinary polynomial approximations. For this problem, the log-log approximations do better than the ordinary Taylor series. The difference in their first-order expansions is slight, but the difference is substantial at order 4. Also, it is clear from Figures 1 and 2 that the order 12 log-log approximation is excellent, implying that the Euler equation error is indistinguishable from machine zero for $k \in [0.6, 1.5]$.

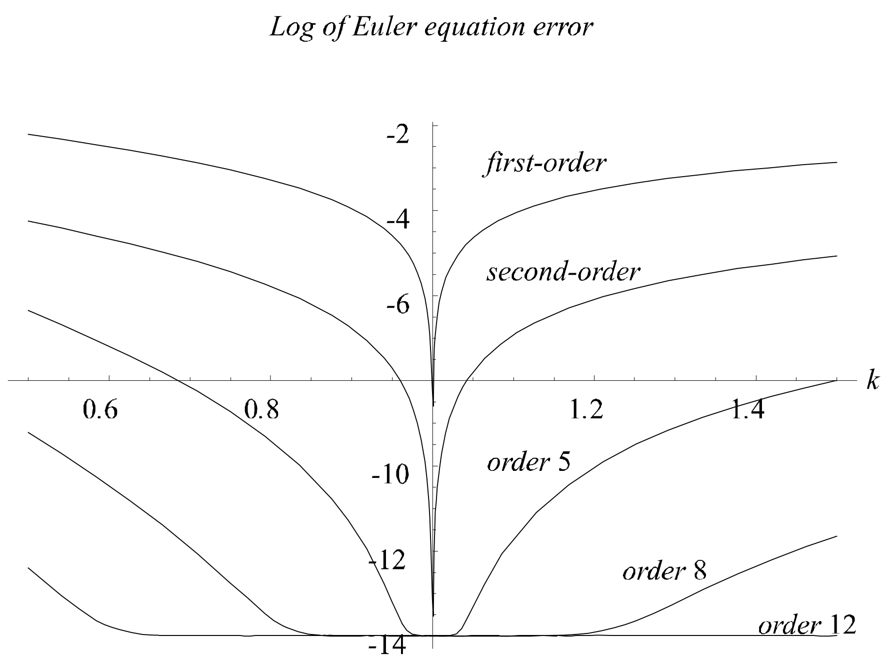

Figure 2: Errors with log-log Taylor series

Table 2: Ordinary versus log-log expansions

$E^{\infty}(.5, 1.5)$

Order	Ordinary polynomial	log-log
1	-1.25	-1.50
2	-1.50	-3.29
3	-1.72	-3.92
4	-1.92	-4.50

2.5. Errors Using Power-Power Changes of Variables. We will now examine Taylor series using the power functions. We first look at the case where $\alpha = \gamma$; we call these $\alpha - \alpha$ approximations. This implies that the consumption function is approximated by

$$C(k) \doteq C(k^*)\left(1 - \eta_1\left(1 - \left(\frac{k}{k^*}\right)^{\alpha}\right)\right)^{1/\alpha}$$

where the elasticity of consumption at the steady state is

$$\eta_1 = \frac{k^* C'(k^*)}{C(k^*)}$$

Figure 3 displays the errors from first-order $\alpha - \alpha$ expansions.

Table 3 compares four different approximations. We first note that the ordinary polynomial and log-log approximations are just special cases of the power function approximations. Table 3 repeats the errors from Tables 1 and 2. We also display the first-order approximation in those cases. We then found the best $\alpha - \alpha$ approximation by finding the minimum of the function in Figure 3. That happens at $\alpha = .306$. Table 3 reports that the error is reduced to -4.03, as displayed in Figure 3, at $\alpha = .306$. Table 3 also displays the first-order approximation in that case.

The next to last row in Table 3 reports the best $\alpha - \gamma$ approximation; it reduces the error by another third of an order of magnitude. However, it is unclear if it is worth the effort to find this solution since the Euler equation error is not a smooth function of (α, γ). Extensive search was necessary to find the global solution (if indeed we did find it). For this problem, it would be less costly to increase the order of the approximation to get that extra accuracy.

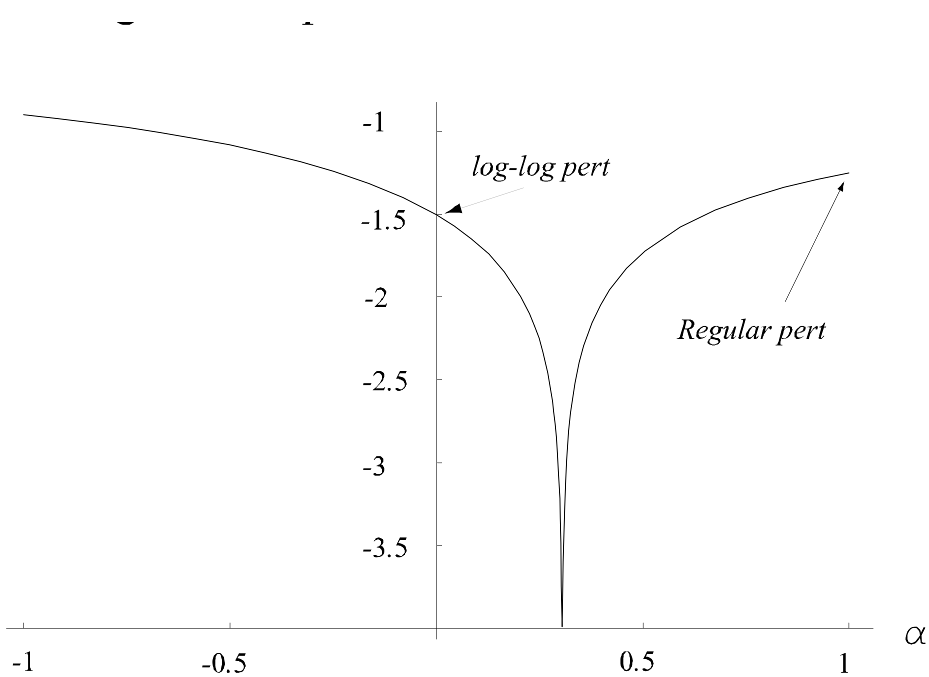

Figure 3: Maximum error with $\alpha - \alpha$ Taylor series

The other remarkable comparison is how well the power function COVs do relative to ordinary methods. The last row of Table 3 repeats the results for the fourth-order log-log expansion. Note that the first-order expansion with the best $\alpha - \gamma$ is practically as good as the conventional fourth-order log-log expansion, and that the best $\alpha - \alpha$ COV is also nearly as good. The improvement is not just due to adding an extra degree of freedom, but comes from a flexible ability to add the right kind of nonlinearity to the problem.

Table 3: First-order approximations

	(α, γ)	$E^{\infty}(.5, 1.5)$	Approximation
Ordinary polynomial	$(1, 1)$	-1.25	$4/19 + 0.116(x-1)$
log-log	$(0, 0)$	-1.50	$4x^{0.552}/19$
best $\alpha - \alpha$	$(0.306, 0.306)$	-4.03	$(0.62 + .343\,(x^{0.306} - 1))^{3.268}$
best $\alpha - \gamma$	$(0.277, 0.250)$	-4.40	$(.677 + .338\,(x^{0.277} - 1))^{4}$
fourth-order log-log	$(0, 0)$	-4.50	

We next consider second-order expansions. We first examine the simple $\alpha-\alpha$ expansions. Figure 4 displays the graph of the Euler equation error over α between -1 and 1. The results here are also striking. First, the log-log approximation is much better than ordinary quadratic expansion. However, there are now two local minima between the log and ordinary cases. Furthermore, both of these minima are about two-orders of magnitude better than log-log second-order expansion.

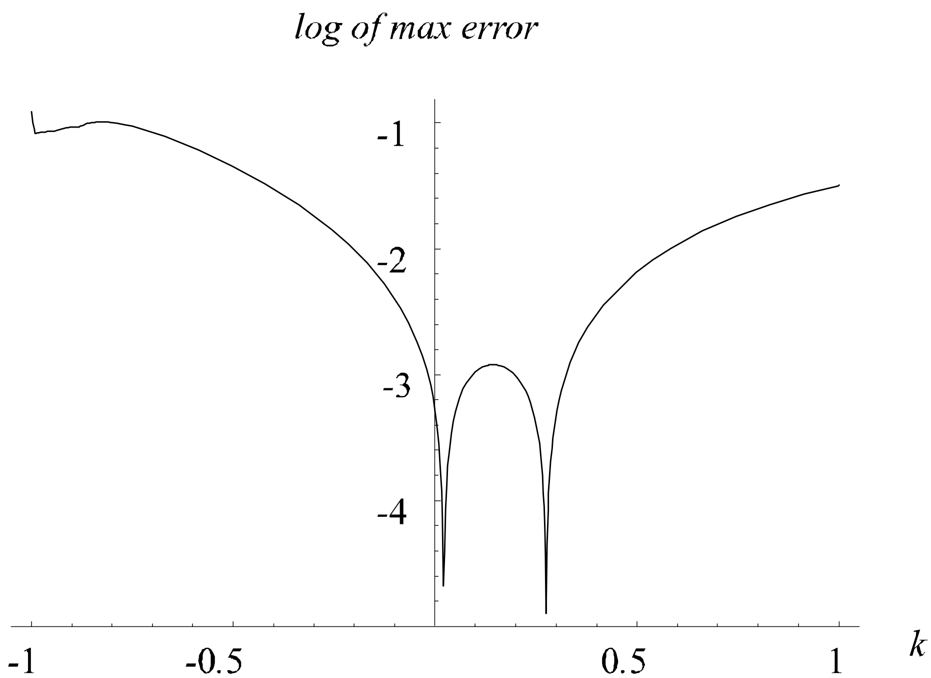

Figure 4: Second-order $\alpha-\alpha$ approximations

Table 4 reports the major results for second-order approximations. Again we see that the optimal $\alpha-\gamma$ approximation is only slightly better than the best $\alpha-\alpha$ approximation.

Table 4: Second-order approximations

	(α, γ)	$E^{\infty}(.5, 1.5)$
Ordinary polynomial	$(1, 1)$	-1.50
log-log	$(0, 0)$	-3.29
best $\alpha - \alpha$	$(0.0228, 0.0228)$	-5.21
best $\alpha - \gamma$	$(0.01059, 0.00004)$	-5.33

In both cases we find that a simple one-dimensional search over various $\alpha - \alpha$ COVs produced substantially better approximations than either the regular or log-log Taylor series. We also found that the global optimum in (α, γ) space added little accuracy but added much to the computational cost of the algorithm. Therefore, it appears that the best approach is to try some simple searches near the ordinary and log-log cases and quit when the returns to further search appear to be diminishing. Remember: the goal is to spend little effort to find something better than the common expansions, not to expend much effort to find the absolute best COV.

Figures 5 and 6 display the graphs for $\alpha - \alpha$ COVs used in third- and fourth-order approximations.

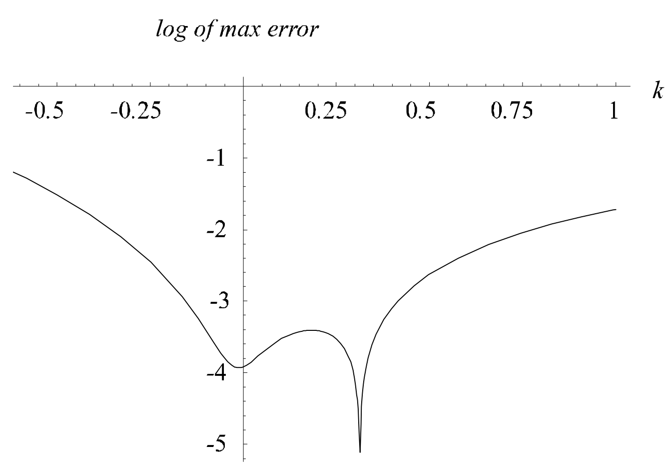

Figure 5: Third-order $\alpha - \alpha$ expansions

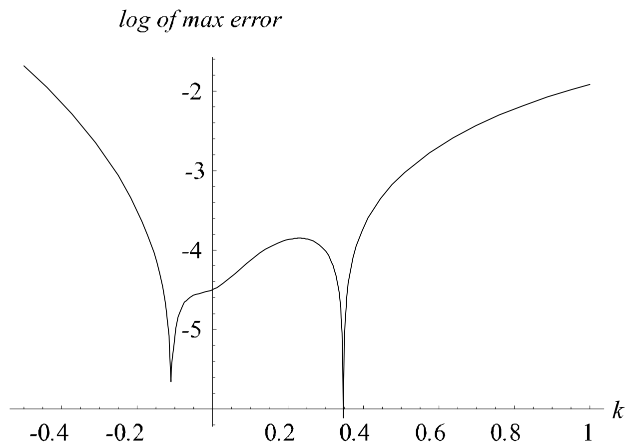

Figure 6: Fourth-order $\alpha - \alpha$ expansions

3. Multidimensional Changes of Variables

These ideas can be directly extended to the case of multivariate functions. Suppose that $x, y \in R^n$. Then a multivariate change of variables is

$$\begin{aligned} y &= Y(x) \\ x &= X(y) \end{aligned}$$

where Y is invertible over the range of x under consideration. A function $f(x)$ can be expressed in the forms

$$\begin{aligned} f(x) &= g(Y(x)) \\ g(y) &= f(X(y)) \end{aligned}$$

Suppose we want to approximate f in a neighborhood of $x = a$. Let $b = Y(a)$. Then, using tensor notation, the derivatives of g are

$$\begin{aligned}
g_\alpha &= f_i X^i_\alpha \\
g_{\alpha\beta} &= f_{ij} X^i_\alpha X^j_\beta + f_i X^i_{\alpha\beta} \\
g_{\alpha\beta\gamma} &= f_{ijk} X^i_\alpha X^j_\beta X^k_\gamma + f_{ij} X^i_{\alpha\gamma} X^j_\beta + f_{ij} X^i_\alpha X^j_{\beta\gamma} \\
&\quad + f_{ik} X^k_\gamma X^i_{\alpha\beta} + f_i X^i_{\alpha\beta\gamma} \\
&= f_{ijk} X^i_\alpha X^j_\beta X^k_\gamma + f_{ij}\left(X^i_{\alpha\gamma} X^j_\beta + X^i_{\alpha\beta} X^j_\gamma + X^i_{\beta\gamma} X^j_\alpha\right) + f_i X^i_{\alpha\beta\gamma}
\end{aligned}$$

The Taylor series of g near $y = b$ is

$$\begin{aligned}
g(y) &\doteq g(b) + g_\alpha (y^\alpha - b^\alpha) \\
&\quad + \frac{1}{2} g_{\alpha\beta} (y^\alpha - b^\alpha)\left(y^\beta - b^\beta\right) \\
&\quad + \frac{1}{6} g_{\alpha\beta\gamma} (y^\alpha - b^\alpha)\left(y^\beta - b^\beta\right)(y^\gamma - b^\gamma)
\end{aligned}$$

and the approximation for f near $x = a$ is

$$\begin{aligned}
f(x) &= g(Y(x)) \doteq g(b) + g_\alpha(b)\left(Y^\alpha(x) - b^\alpha\right) \\
&\quad + \frac{1}{2} g_{\alpha\beta}(b)\left(Y^\alpha(x) - b^\alpha\right)\left(Y^\beta(x) - b^\beta\right) \\
&\quad + \frac{1}{6} g_{\alpha\beta\gamma}(b)\left(Y^\alpha(x) - b^\alpha\right)\left(Y^\beta(x) - b^\beta\right)\left(Y^\gamma(x) - b^\gamma\right)
\end{aligned}$$

More generally, we transform both the range and domain of f. That is, we find $h : R \to R$ and $Y : R \to R$ such that

$$\begin{aligned}
h(f(x)) &= g(Y(x)) \\
g(y) &= h(f(X(y)))
\end{aligned}$$

The derivatives of g are

$$\begin{aligned}
g_\alpha &= h_A f^A_i X^i_\alpha \\
g_{\alpha\beta} &= h_{AB} f^B_j X^j_\beta f^A_{ij} X^i_\alpha + h_A f^A_{ij} X^j_\beta X^i_\alpha + h_A f^A_i X^i_{\alpha\beta}
\end{aligned}$$

the Taylor series of g is the multivariate polynomial

$$\begin{aligned} g^{Tay}(b) &\doteq g(b) + g_\alpha (y^\alpha - b^\alpha) \\ &+\frac{1}{2} g_{\alpha\beta} (y^\alpha - b^\alpha)\left(y^\beta - b^\beta\right) \end{aligned}$$

and the approximation of f is

$$f(x) \doteq h^{-1}\left(g^{Tay}(Y(x))\right)$$

Again,we see that the coefficients of the expansion of f in terms of the new variables y are easily computed once we know the ordinary Taylor series for f.

Normally one would consider simple COVs where a variable y_i depends on only one component of x. In that case, the formulas above involve only a small amount of computation, particularly for the first- and second-order expansions.

4. CONCLUSION

Perturbation methods are becoming increasingly useful in economic analysis. However, the marginal cost of computing higher-order terms can be rather high. This paper shows how to better use the information produced by perturbation methods. The key idea is to consider a variety of expansions which are all locally equivalent but may differ globally, and then pick the one which does best in terms of minimizing overall error. We have shown that this can increase accuracy by two orders of magnitude in a simple problem. This shows that one can, at little computational cost, use the information from perturbation methods to substantially improve the quality of the final approximation.

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