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CS 530

Probabilistic and Geometric Methods

Fourier Transforms

Complex Inner Product

- We need to use the complex conjugate

$$\langle a, b \rangle = \sum_i a_i^* b_i \qquad \langle a, b \rangle = \int_{-\infty}^{\infty} a(s)^* b(s) ds$$

Shah Function

- A "train" of impulse functions

$$\mathrm{III}(t) = \sum_{n=-\infty}^{\infty} \delta(t-n)$$

- 2-pi periodic

$$\frac{1}{2\pi}\mathrm{III}\left(\frac{t}{2\pi}\right) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \delta\left(\frac{t}{2\pi}-n\right)$$

A Shah Basis

- Consider 2-pi periodic shah functions:

$$\frac{1}{2\pi}\mathbf{III}\left(\frac{t-\tau}{2\pi}\right)$$

- What’s the inner product for 2 taus?

A Shah Basis

- How do we represent a vector in a new basis?

A Shah Basis

- How do we represent a vector in a new basis?
  - Using Inner products
  - We want this:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} w(t)\,\mathrm{III}\left(\frac{t-\tau}{2\pi}\right)d\tau$$

  - *w(t)* is a coefficient in the shah basis
  - Note: *f(t)* is 2pi periodic

A Shah Basis

- So we want this:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} w(t)\,\mathbf{III}\left(\frac{t-\tau}{2\pi}\right)d\tau$$

- Just like vectors, we get coefficients using inner products

$$w(t) = \left\langle \frac{1}{2\pi}\mathbf{III}\left(\frac{t-\tau}{2\pi}\right), f(t) \right\rangle = \int_{-\infty}^{\infty} f(t)\,\mathbf{III}\left(\frac{t-\tau}{2\pi}\right)d\tau$$

A Shah Basis

- We can do a rewrite:

$$\int_{-\infty}^{\infty} f(t)\frac{1}{2\pi}\mathbf{III}\left(\frac{t-\tau}{2\pi}\right)d\tau = \int_{-\pi}^{\pi} f(t)\delta(t-\tau)d\tau$$

- Which is just

$$\int_{-\pi}^{\pi} f(t)\delta(t-\tau)d\tau = f(t) = w(t)$$

- So, f(t) is the coefficient in the Shah basis

Back to Harmonic Signals

- We can represent frequency as "angular frequency"

$$\omega = 2\pi s \qquad s = \frac{\omega}{2\pi}$$

$$e^{i2\pi st} = e^{i\omega t}$$

Harmonic Signals

- “length” over 2pi interval

$$\left| e^{i\omega t} \right| = \left\langle e^{i\omega t}, e^{i\omega t} \right\rangle^{1/2} = \int_{-\pi}^{\pi} e^{-i\omega t} e^{i\omega t} d\tau$$

Harmonic Functions

- What is the inner product between 2 harmonic functions?

$$\left\langle e^{i\omega_1 t}, e^{i\omega_2 t} \right\rangle^{1/2} = \int_{-\pi}^{\pi} e^{-i\omega_1 t} e^{i\omega_2 t} d\tau = \left. \frac{e^{i(\omega_2 - \omega_1)t}}{i(\omega_2 - \omega_1)} \right]_{-\pi}^{\pi}$$

- …if omega_1 and omega_2 are integers?

Fourier Basis

- If integer frequencies are orthogonal, we have a basis for representing functions

$$f(t) = \frac{1}{\sqrt{2\pi}} \sum_{\omega=-\infty}^{\infty} F(\omega) e^{i\omega t}$$

- What are the F(omega) coefficients?

$$F(\omega) = \left\langle \frac{1}{\sqrt{2\pi}} e^{-i\omega t}, f(t) \right\rangle = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(t) e^{-i\omega t} dt$$

Fourier Basis for Shah function

- Recall $$\frac{1}{2\pi}\mathbf{III}\left(\frac{t}{2\pi}\right)$$

Fourier Basis

- What if f isn't periodic?
  - If it's "square integrable"
  - …then it it has a finite square integral

$$\int_{-\infty}^{\infty} f^*(t) f(t) dt < \infty$$

- Then it's fourier transform is

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

What have we learned?

- We can factor function into (potentially infinite) bases.
- The harmonic functions are one such basis.
- The coefficients in this basis are the fourier transform

What have we learned?

- When the function is periodic, the representation in the basis is a series
- When the function is no periodic it's an integral
- The "standard basis" for a function is the Shah basis

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