CS 261
Mathematical Foundations of Computer Science

Chapter 3
Counting

Chapter 3 Quiz

- Monday, June 27 2005.
- Approximately 30 minutes.
- One 8½ x 11 inch page of notes – both sides (no books).
- No Calculators: you will need to be able to simplify: i.e.

$$\frac{25!}{5!(23!)} = \frac{25 \cdot 24 \cdot 23!}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 23!} = \frac{25 \cdot 24}{5 \cdot 4 \cdot 3 \cdot 2} = \frac{5(5) \cdot 4(3)(2)}{5 \cdot 4 \cdot 3 \cdot 2} = 5$$

Coverage:

- 3.1: #2, #3, #8, #9, #11, #12, #19, #21, #22, #28, #31
- 3.2: #1, #6, #9, #21, #24, #27, #28, #33
- 3.3: #4, #5, #6, #11, #18
- 3.4: #9, #10, #12, #13, #15, #20, #21, #37, #39, #40
- 3.5: #1, #2, #3, #4, #5, #6, #9, #10, #18, #19, #20, #21, #22

Solving Recurrence Relation into an Explicit Formula

Linear homogeneous relation of degree k:

$$a_n = r_1 a_{n-1} + r_2 a_{n-2} + \ldots + r_k a_{n-k}$$ where r_i is a constant

Linear homogeneous relation of degree 2:

$$a_n = r_1 a_{n-1} + r_2 a_{n-2}$$

Characteristic Equation: $x^2 - r_1 x - r_2 = 0$

Two distinct roots $\{s_1, s_2\}$: $a_n = u s_1^n + v s_2^n$

Single root $\{s\}$ $a_n = u s^n + v n s^n$

Ex: 3.5 #18: $a_n = 4a_{n-1} + 5a_{n-2}$, $a_1=2$, $a_2=6$

$a_n = r_1 a_{n-1} + r_2 a_{n-2}$ $x^2 - r_1 x - r_2 = 0$ $a_n = u(s_1)^n + v(s_2)^n$

$x^2 - 4x - 5 = 0$

$(x - 5)(x + 1) = 0 \quad \Rightarrow$ roots $\{s_1, s_2\}$: $\{5, -1\}$

$a_1 = 2 = u(5)^1 + v(-1)^1$	$a_2 = 6 = u(5)^2 + v(-1)^2$
$2 = 5u - v$	$6 = 25u + v$
$v = 5u - 2$	$v = 6 - 25u$

$5u - 2 = 6 - 25u$

$u = 8/30 = 4/15$

$v = 5u - 2 = 5(4/15) - 2 = -2/3$

$$a_n = \frac{4}{15}5^n - \frac{2}{3}(-1)^n$$

Check:

$$a_3 = \frac{4}{15}5^3 - \frac{2}{3}(-1)^3 = \frac{4\cdot 125}{15} + \frac{2}{3} = \frac{4\cdot 25}{3} + \frac{2}{3} = \frac{102}{3} = 34$$

Quadratic Equation verses Factoring

For Ex: 3.5 #18, we found the roots of $x^2 - 4x - 5 = 0$

by factoring with intuition:

$(x - 5)(x + 1) = 0 \quad \Rightarrow \quad$ roots $\{s_1, s_2\}$: $\{5, -1\}$

If the factors cannot be found by inspection, use the quadratic equation:

$$ax^2 + bx + c = 0 \quad \Rightarrow \quad \frac{b \pm \sqrt{b^2 - 4ac}}{-2a}$$

$$= \frac{-4 \pm \sqrt{(-4)^2 - 4(1)(-5)}}{-2(1)}$$

$$= \frac{-4 \pm \sqrt{16+20}}{-2} \quad = \quad \frac{-4 \pm \sqrt{36}}{-2} \quad = \quad \frac{-4 \pm 6}{-2}$$

$$= \left\{\frac{-4+6}{-2}, \quad \frac{-4-6}{-2}\right\} = \{-2/2, \quad -10/-2\} = \{-1, \quad 5\}$$

Ex: 3.5 #19: $b_n = -3b_{n-1} - 2b_{n-2}, \quad b_1 = -2, \quad b_2 = 4$

$b_n = r_1 b_{n-1} + r_2 b_{n-2} \qquad x^2 - r_1 x - r_2 = 0 \qquad b_n = u(s_1)^n + v(s_2)^n$

$x^2 + 3x + 2 = 0$

$(x + 2)(x + 1) = 0 \quad \Rightarrow \quad$ roots $\{s_1, s_2\}$: $\{-1, -2\}$

$b_1 = -2 = u(-1)^1 + v(-2)^1 \qquad b_2 = 4 = u(-1)^2 + v(-2)^2$

$-2 = -u - 2v \qquad 4 = u + 4v$

$u = 2 - 2v \qquad u = 4 - 4v$

$2 - 2v = 4 - 4v$

$v = 1$

$u \; = \; 2 - 2v \; = \; 2 - 2(1) \; = \; 0$

$$b_n = 0(-1)^n + 1(-2)^n \quad = \quad (-2)^n$$

Check:

$$b_3 = (-2)^3 = -8$$

Ex: 3.5 #20: $c_n = -6c_{n-1} - 9c_{n-2}$, $c_1 = 2.5$, $c_2 = 4.7$

$c_n = r_1 c_{n-1} + r_2 c_{n-2}$ $\quad x^2 - r_1 x - r_2 = 0$ $\quad c_n = u(s)^n + v(n)(s)^n$

$x^2 + 6x + 9 = 0$

$(x + 3)(x + 3) = 0 \quad \Rightarrow \quad$ root $\{s\}$: $\{-3\}$

$c_1 = 2.5 = u(-3)^1 + v(1)(-3)^1$

$2.5 = (-3u - 3v)$

$3v = -3u - 2.5$

$18v = -18u - 15$

$c_2 = 4.7 = u(-3)^2 + v(2)(-3)^2$

$4.7 = 9u + 18v$

$18v = 4.7 - 9u$

$-18u - 15 = 4.7 - 9u$

$u = -(19.7)/9$

$v = -3u - 2.5 = -3[-(19.7)/9] - 2.5 = (12.2)/9$

$$c_n = -\frac{19.7}{9}(-3)^n + \frac{12.2}{9}(n)(-3)^n$$

Explicit Formula for Fibonacci Sequence

The recurrence relation for the Fibonacci sequence is:

$f_n = f_{n-1} + f_{n-2}$, $\quad f_1 = 1$, $f_2 = 1$

Which gives the sequence:

1, 1, 2, 3, 5, 8, 13, 21, 34, ...

Example 9, derives the explicit formula to be:

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n \approx \frac{(1.618)^n}{\sqrt{5}} - \frac{(-0.618)^n}{\sqrt{5}}$$

Evaluate the explicit formula for $n = 7$.

$$f_7 \approx \frac{(1.618)^7}{\sqrt{5}} - \frac{(-0.618)^7}{\sqrt{5}} \approx \frac{29.03}{2.2361} - \frac{-0.03443}{2.2361} \approx 12.98 + 0.015 = 12.995$$

What can be said about the two terms as n gets large?

Work as Study Group at Board

- 3.5: # 21, #22
- 3.4: #12, #13, #15
- 3.3: #18
- 3.2: #21, #24
- 3.1: #2, #3, #12