Lattice Laws Forcing Distributivity Under Unique Complementation

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Abstract

In this paper, we give several new lattice identities valid in non-modular lattices such that a uniquely complemented lattice satisfying any of these identities is necessarily Boolean. Since some of these identities are consequences of modularity as well, these results generalize the classical result of Birkhoff and von Neumann that every uniquely complemented modular lattice is Boolean. In particular, every uniquely complemented lattice in $M \vee N_5$, the least non-modular variety, is Boolean.

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1 Introduction

In 1904 Huntington [6] conjectured that every uniquely complemented lattice must be distributive (and hence a Boolean algebra). In 1945, R. P. Dilworth shattered this conjecture by proving [4] that every lattice can be embedded in a uniquely complemented lattice. For a much strengthened version of the same resuslt, see Adams and Sichler [1].

However, it turns out that if we assume almost any mild "natural" condition on a uniquely complemented lattice, then it becomes distributive. As an example, we have the theorem of Birkhoff and von Neumann that every uniquely complemented modular lattice is Boolean. Other conditions which force a uniquely complemented lattice to be distributive include weak atomicity (Bandelt and Padmanabhan [3]) and upper continuity (Bandelt [2] and Salii [?], [?] independently). A monograph written by Salii [?] gives a comprehensive survey of results of this type. Among these mild additional conditions which force a uniquely complemented lattice to be distributive, modularity is the only known condition which is a lattice identity. In this paper, we give a number of new lattice identities valid in non-modular lattices such that a uniquely complemented lattice satisfying any of these identities is necessarily Boolean. In particular, the least non-modular variety $M \vee N_5$, turns out to be one such variety.

Following [7], we call a variety K of lattices a *Huntington variety* if every uniquely complemented member of K is distributive. For example, the Birkhoff-von Neumann theorem mentioned above says that the modular varietry is Huntington.

Theorem 1. *The variety of lattices defined by*

$$(x \wedge (y \vee (x \wedge z))) \vee (x \wedge (z \vee (x \wedge y))) = x \wedge (z \vee y)$$

is a non-modular Huntington variety.

Proof. We show that the condition $a \wedge b' = 0$ forces the inequality $a \leq b$ and hence by a well-known theorem of O. Frink (e.g. see [8]), the lattice will necessarily be Boolean. Indeed, let $a \wedge b' = 0$ for some two elements a, b in a uniquely complemented lattice satisfying the identity

$$(x \wedge (y \vee (x \wedge z))) \vee (x \wedge (z \vee (x \wedge y))) = x \wedge (z \vee y).$$

Put $z = x'$ in the above to get

$$(x \wedge y) \vee (x \wedge (x' \vee (x \wedge y))) = x \wedge (x' \vee y).$$

Now let $x = b'$, $y = a$. We have

$$(b' \wedge a) \vee (b' \wedge (b \vee (b' \wedge a))) = b' \wedge (b \vee a).$$

So if we assume that $a \wedge b' = 0$, then we get $b' \wedge (b \vee a) = 0$. Also, $b' \vee (b \vee a) = (b' \vee b) \vee a = 1 \vee a = 1$. Thus both b and $b \vee a$ are complements of the element

b'. Since the lattice is uniquely complemented, we get the desired conclusion $b \vee a = b$. In other words, we have proved that the given lattice satisfies the bi-implication $a \leq b$ if and only if $a \wedge b' = 0$ and hence, by Frink's theorem, the lattice is distributive. $\square$

It is interesting to note that Otter, a first-order theorem prover obtained verbatically a similar proof (see Appendix).

Theorem 2. *The self-dual variety of lattices defined the pair of equations*

$$x \vee (y \wedge (x \vee z)) = x \vee (y \wedge (z \vee (x \wedge (z \vee y))))$$
$$x \wedge (y \vee (x \wedge z)) = x \wedge (y \vee (z \wedge (x \vee (z \wedge y))))$$

is Huntington.

The proof is in the Appendix.

2 Non-Modular Huntington Equations

Given a uniquely complemented lattice, if any of the following equations holds, the lattice is a Boolean algebra.

$$x \wedge (y \vee (z \wedge (x \vee (z \wedge u)))) = x \wedge (y \vee (z \wedge (x \vee u))) \tag{H01}$$
$$x \wedge (y \vee (z \wedge ((x \wedge (y \vee z)) \vee (y \wedge z)))) = x \wedge (y \vee (x \wedge z)) \tag{H02}$$
$$x \wedge ((x \wedge (y \vee (x \wedge z))) \vee (z \wedge (x \vee y))) = x \wedge (y \vee (x \wedge z)) \tag{H06}$$
$$x \wedge ((x \wedge y) \vee ((x \wedge z) \vee (y \wedge (x \vee z)))) = (x \wedge y) \vee (x \wedge z) \tag{H22}$$
$$x \wedge ((y \wedge (x \vee (z \wedge u))) \vee (z \wedge u)) = x \wedge ((x \wedge y) \vee (z \wedge u)) \tag{H32}$$
$$x \wedge (y \vee ((x \wedge z) \vee (z \wedge (y \vee u)))) = x \wedge (y \vee (z \wedge (x \vee u))) \tag{H56}$$
$$x \wedge (y \vee ((x \wedge z) \vee (z \wedge u))) = x \wedge (y \vee (z \wedge (x \vee u))) \tag{H58}$$
$$x \wedge (y \vee ((x \vee y) \wedge (z \vee (x \wedge y)))) = (x \wedge y) \vee (x \wedge (y \vee z)) \tag{H62}$$
$$x \wedge (y \vee ((y \vee z) \wedge (u \vee (y \wedge z)))) = x \wedge ((y \vee z) \wedge (y \vee u)) \tag{H70}$$
$$x \wedge (y \wedge (z \vee (x \wedge (u \vee (x \wedge z))))) = x \wedge (y \wedge (z \vee u)) \tag{H73}$$
$$x \wedge (y \vee (z \wedge (u \vee (x \wedge y)))) = x \wedge (y \vee (z \wedge (y \vee u))) \tag{H83}$$
$$x \wedge ((x \wedge (y \vee (x \wedge z))) \vee (z \wedge u)) = x \wedge (y \vee (z \wedge (x \vee u))) \tag{H86}$$

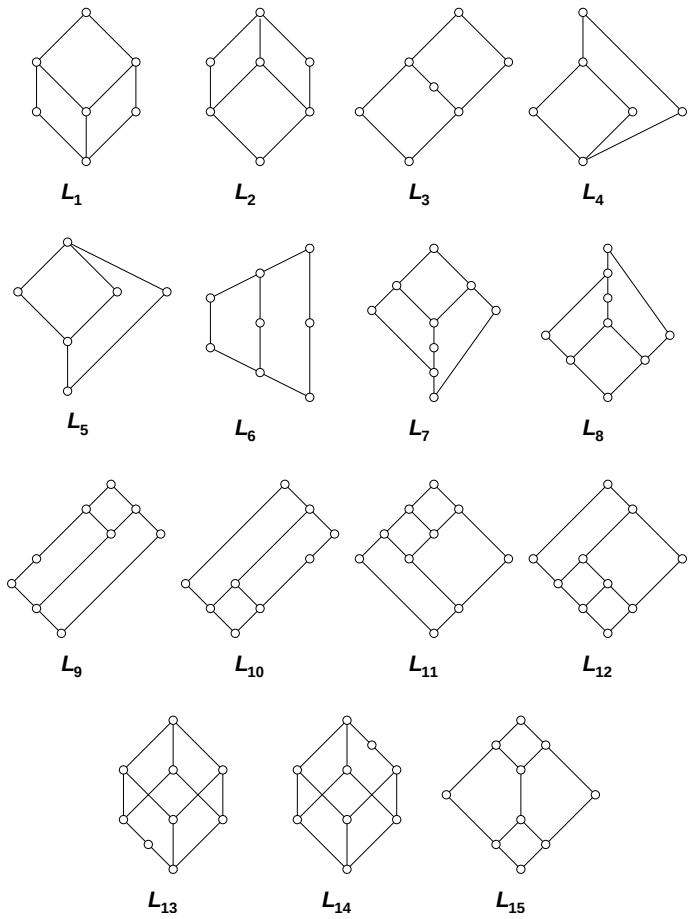

Figure 1: All the Covers of the Least Non-Modular Lattice N_5

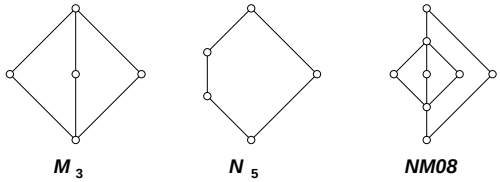

Figure 2: *what should this caption be?*

Table 1: Lattices (L_1 – L_{15}, M_3, N_5, $NM08$) for which the Equations Hold

(H01)	1	2				7			10	11		13	14	15		N_5	
(H02)	1		4		6	7	8	9	10	11	12	13	14	15	M_3	N_5	$NM08$
(H06)	1		4		6	7	8	9	10	11		13	14	15	M_3	N_5	$NM08$
(H22)	1		4		6	7		9	10	11		13		15	M_3	N_5	$NM08$
(H32)	1				6	7	8	9	10	11		13	14	15		N_5	
(H56)	1				6	7		9	10	11		13	14	15		N_5	
(H58)	1					7			10	11		13	14	15		N_5	
(H62)		2		5	6	7	8	9	10		12	13	14	15	M_3	N_5	$NM08$
(H70)		2		5	6		8	9	10		12		14	15	M_3	N_5	$NM08$
(H73)		2			6	7	8	9	10		12	13	14	15		N_5	
(H83)					6	7	8	9	10			13	14	15		N_5	
(H86)						7			10			13	14	15		N_5	

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Appendix

Proof of Theorem 1

Proof of Theorem 2