LATTICE LAWS FORCING DISTRIBUTIVITY UNDER UNIQUE COMPLEMENTATION

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Abstract. We give several new lattice identities valid in nonmodular lattices such that a uniquely complemented lattice satisfying any of these identities is necessarily Boolean. Since some of these identities are consequences of modularity as well, these results generalize the classical result of Birkhoff and von Neumann that every uniquely complemented modular lattice is Boolean. In particular, every uniquely complemented lattice in $M \vee \mathcal{V}(N_5)$, the least nonmodular variety, is Boolean.

1. Introduction

In 1904 Huntington [4] conjectured that every uniquely complemented lattice must be distributive (and hence a Boolean algebra). In 1945, R. P. Dilworth shattered this conjecture by proving [2] that every lattice can be embedded in a uniquely complemented lattice. For a much powerful version of the same results, see Adams and Sichler [1].

In spite of these deep results, it is still hard to find "nice" examples of uniquely complemented lattices that are not Boolean. The reason is that uniquely complemented lattices having a little extra structure most often turn out to be distributive. This seems to be the essence of Huntington's conjecture. For example, we have the theorem of Garrett Birkhoff and von Neumann that every uniquely complemented modular lattice is Boolean. Following [10], we call a lattice property P a *Huntington property* if every uniquely complemented P-lattice is distributive. Similarly, a lattice variety K is said to be a *Huntington variety* if every uniquely complemented lattice in K is Boolean. In this terminology, the modular lattices are the largest previously known Huntington variety. A monograph by Salii [13] gives a comprehensive survey of known Huntington properties. Among these, modularity is the only known condition that is a lattice identity. In this paper, we give a number of new nonmodular Huntington varieties, any of which can be construed as a generalization of the von Neumann-Birkhoff theorem.

The automated theorem provers Otter [6] and Prover9 [8], and the program Mace4 [7], which searches for finite algebras, were used in this work. Several automated proofs are given in the appendix. The Web page associated with this paper [12] contains additional Huntington identities and automated proofs supporting this work.

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2. A Nonmodular Huntington Variety

Here we give a lattice identity that defines a nonmodular Huntington variety. Several others are given in the following sections and in the supporting Web page [12].

Theorem 1. *The variety of lattices defined by*

$$(x \wedge (y \vee (x \wedge z))) \vee (x \wedge (z \vee (x \wedge y))) = x \wedge (z \vee y) \tag{H69}$$

is a nonmodular Huntington variety.

Proof. We show that the condition $a \wedge b' = 0$ forces the inequality $a \leq b$ and hence by a well-known theorem of O. Frink [11], the lattice will necessarily be Boolean. Indeed, let $a \wedge b' = 0$ for some two elements a, b in a uniquely complemented lattice satisfying the identity

$$(x \wedge (y \vee (x \wedge z))) \vee (x \wedge (z \vee (x \wedge y))) = x \wedge (z \vee y).$$

Put $z = x'$ in the above to get

$$(x \wedge y) \vee (x \wedge (x' \vee (x \wedge y))) = x \wedge (x' \vee y).$$

Now let $x = b'$, $y = a$. We have

$$(b' \wedge a) \vee (b' \wedge (b \vee (b' \wedge a))) = b' \wedge (b \vee a).$$

If we assume that $a \wedge b' = 0$, then we get $b' \wedge (b \vee a) = 0$. Also, $b' \vee (b \vee a) = (b' \vee b) \vee a = 1 \vee a = 1$. Thus both b and $b \vee a$ are complements of the element b'. Since the lattice is uniquely complemented, we get the desired conclusion $b \vee a = b$. In other words, we have proved that the given lattice satisfies the bi-implication $a \leq b$ if and only if $a \wedge b' = 0$. Hence, by Frink's theorem, the lattice is distributive. □

3. Huntington Implications

Here we show Huntington properties that are implications. These can be used, among other purposes, to show that lattice identities are Huntington.

Theorem 2. (See [10].) *A uniquely complemented lattice satisfying any one of the following three implications (or their duals) is distributive.*

(SD-$\vee$) $\quad x \vee y = x \vee z \Rightarrow x \vee y = x \vee (y \wedge z)$

(CD-$\vee$) $\quad x \vee y = x \vee z \Rightarrow (x \wedge y) \vee (x \wedge z) = x \wedge (y \vee z)$

(CM-$\vee$) $\quad x \vee y = x \vee z \Rightarrow x \wedge ((x \wedge y) \vee z) = (x \wedge y) \vee (x \wedge z)$

A proof of (CD-$\vee$) is given in the appendix. Proofs of the other two cases are given on the supporting Web page [12].

Corollary 1. *A uniquely complemented lattice satisfying the identity*

$$x \wedge ((y \wedge (x \vee z)) \vee (z \wedge (x \vee y))) = (x \wedge y) \vee (x \wedge z) \tag{H82}$$

is Huntington.

Proof. It is easy to see that (H82) implies the lattice implication (CD-$\vee$). Indeed, if

$$x \vee y = x \vee z,$$

then

$$\begin{aligned}(x \wedge y) \vee (x \wedge z) &= x \wedge ((y \wedge (x \vee z)) \vee (z \wedge (xvy))) && \text{by (H82)}\\ &= x \wedge ((y \wedge (x \vee y)) \vee (z \wedge (x \vee z))) && \text{by hypothesis}\\ &= x \wedge (y \vee z)\end{aligned}$$

□

As the reader can see, the identity (H82) is designed to show that there are lattice identities that formally imply such implications. Using powerful concepts like the bounded homomorphisms of Ralph McKenzie, one could show that there are many lattice identities like (H82) that formally imply (SD-$\vee$), (SD-$\wedge$), (CD-$\vee$), and so on. In fact, every finite lattice satisfying (SD-$\vee$) or (SD-$\wedge$) will satisfy a lattice identity that formally implies the respective implication, and all these identities are examples of nonmodular Huntington identities (for more details, please see [10]).

Table 1 lists several Huntington identities justified by the preceding Huntington implications. Proofs can be found on the supporting Web page [12]. None of the identities are equivalent (given lattice theory).

Table 1. Huntington Identities Justified by Huntington Implications

Name	Identity	Justification
H18	$(x \wedge y) \vee (x \wedge z) = x \wedge ((x \wedge y) \vee ((x \wedge z) \vee (y \wedge (x \vee z))))$	CM-$\vee$
H50	$x \wedge (y \vee (z \wedge (x \vee u))) = x \wedge (y \vee (z \wedge (x \vee (z \wedge (y \vee u)))))$	SD-$\vee$
H51	$x \wedge (y \vee (z \wedge (x \vee u))) = x \wedge (y \vee ((x \wedge z) \vee (z \wedge u)))$	SD-$\vee$
H64	$x \wedge (y \vee z) = x \wedge (y \vee (x \wedge (z \vee (x \wedge (y \vee (x \wedge z))))))$	SD-$\wedge$
H68	$x \wedge (y \vee z) = x \wedge (y \vee (x \wedge (z \vee (x \wedge y))))$	SD-$\wedge$
H69	$x \wedge (y \vee z) = (x \wedge (z \vee (x \wedge y))) \vee (x \wedge (y \vee (x \wedge z)))$	SD-$\wedge$
H76	$x \wedge (y \vee (z \wedge (y \vee u))) = x \wedge (y \vee (z \wedge (u \vee (x \wedge y))))$	SD-$\vee$, SD-$\wedge$
H79	$x \wedge (y \vee (z \wedge (x \vee u))) = x \wedge ((x \wedge (y \vee (x \wedge z))) \vee (z \wedge u))$	SD-$\vee$, SD-$\wedge$
H80	$(x \wedge y) \vee (x \wedge z) = x \wedge ((x \wedge y) \vee (z \wedge (x \vee (y \wedge (x \vee z)))))$	CM-$\vee$
H82	$(x \wedge y) \vee (x \wedge z) = x \wedge ((y \wedge (x \vee z)) \vee (z \wedge (x \vee y)))$	CD-$\vee$, CM-$\vee$

4. More Huntington Identities

This section contains several nonmodular Huntington identities that do not satisfy the Huntington implications (SD-$\vee$), (CD-$\vee$), (CM-$\vee$), or their duals.

Theorem 3. *The variety of lattices defined by*

$$x \wedge (y \vee z) = x \wedge (y \vee ((x \vee y) \wedge (z \vee (x \wedge y)))) \tag{H58}$$

is a nonmodular Huntington variety.

Proof. (The automatic proof from which this proof was derived is given in the appendix.) We show that any uniquely complemented lattice satisfying (H58) also satisfies the order reversibility property $a \leq b \Rightarrow b' \leq a'$. Assume $a \leq b$; therefore $a \wedge b' = 0$. In (H58), set $x = a$, $y = b'$, and $z = (a \vee b')'$; then simplify the right-hand side, giving

$$a \wedge (b' \vee (a \vee b')') = 0.$$

Unique complementation gives $b' \vee (a \vee b')' = a'$, and therefore $b' \leq a'$. Thus the unary mapping $x \mapsto x'$ is order reversible, and it is well known that this forces distributivity of a uniquely complemented lattice (see [13, p. 48, Cor. 1]; for a computer proof see [12]). □

Along with (H58), additional Huntington identities not satisfying the Huntington implications are shown in the following list. Automated proofs are given on the supporting Web page [12].

(H1) $x \wedge (y \vee (z \wedge (x \vee u))) = x \wedge (y \vee (z \wedge (x \vee (z \wedge u))))$

(H2) $x \wedge (y \vee (x \wedge z)) = x \wedge (y \vee (z \wedge ((x \wedge (y \vee z)) \vee (y \wedge z))))$

(H3) $x \wedge (y \vee (x \wedge z)) = x \wedge (y \vee (z \wedge (y \vee (x \wedge (z \vee (x \wedge y))))))$

(H55) $x \vee (y \wedge (x \vee z)) = x \vee (y \wedge (z \vee (x \wedge (z \vee y))))$

(H58) $x \wedge (y \vee z) = x \wedge (y \vee ((x \vee y) \wedge (z \vee (x \wedge y))))$

5. Covers of N_5

Since we are interested in discovering nonmodular lattice identities that force distributivity under unique complementation, we naturally look at all the covers of the variety $\mathcal{V}(N_5)$. Ralph McKenzie [9] constructed the fifteen lattices L1–L15, shown in Figure 2, whose varieties are join-irreducible covers of the least nonmodular variety $\mathcal{V}(N_5)$. It is a deep result in lattice theory that there are exactly 16 covers of $\mathcal{V}(N_5)$: the fifteen varieties of McKenzie and the trivial $\mathcal{V}(M_3) \vee \mathcal{V}(N_5)$ (Figure 1). The results in this paper demonstrate that all these sixteen varieties are, in fact, Huntington varieties. Table 2 lists the lattices from Figures 2 and 1 for which the Huntington identities given in the paper hold. For more information on these nonmodular lattice laws and other details, see [10].

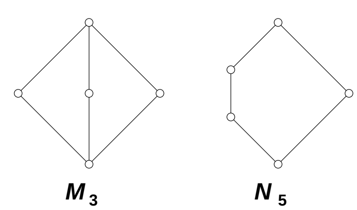

Figure 1. Lattices M_3 and N_5

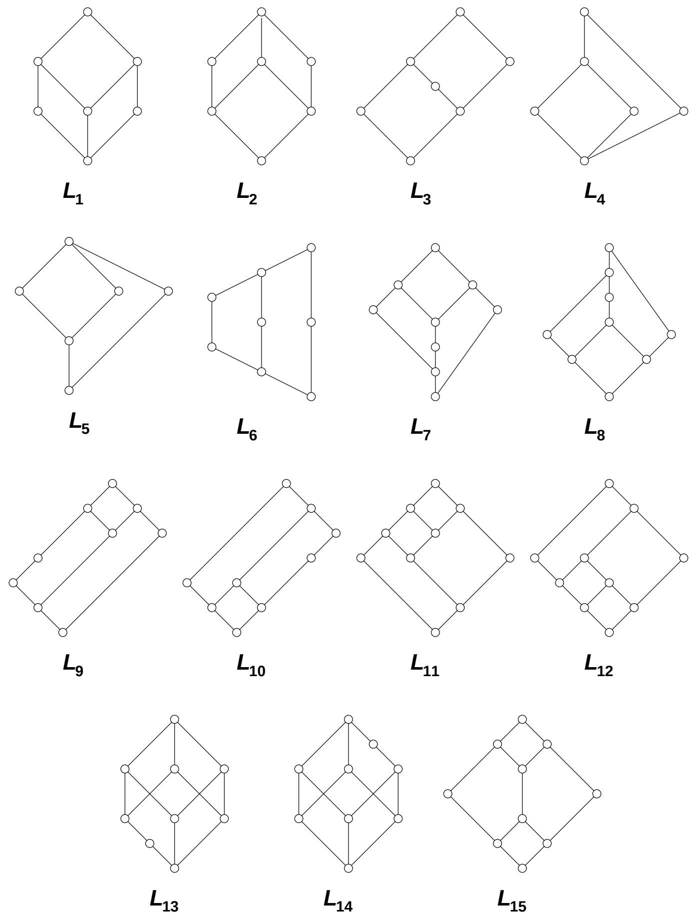

FIGURE 2. All Covers of the Least Nonmodular Lattice N_5

TABLE 2. Lattices ($L_1 - L_{15}$, M_3, N_5) for which the Identities Hold

(H1)	1	2					7			10	11		13	14	15		N_5
(H2)	1			4		6	7	8	9	10	11	12	13	14	15	M_3	N_5
(H3)	1			4		6	7	8	9	10	11	12	13	14	15	M_3	N_5
(H18)	1			4		6	7		9	10	11		13		15	M_3	N_5
(H50)	1					6	7		9	10	11		13	14	15		N_5
(H51)	1						7			10	11		13	14	15		N_5
(H55)		2			5	6	7	8	9	10		12	13	14	15	M_3	N_5
(H58)		2			5	6	7	8	9	10		12	13	14	15	M_3	N_5
(H64)		2				6	7	8	9	10	11	12	13	14	15		N_5
(H68)		2				6	7	8	9	10		12	13	14	15		N_5
(H69)		2				6	7	8	9	10		12	13	14	15		N_5
(H76)						6	7	8	9	10			13	14	15		N_5
(H79)							7			10			13	14	15		N_5
(H80)	1		3	4		6	7	8	9	10	11		13		15	M_3	N_5
(H82)	1			4		6	7		9	10	11		13		15		N_5

6. Methods of Discovery

Two methods were used to find the Huntington identities presented in this paper. The first was to automatically generate a great number of candidates and submit each to an automated theorem-proving program, and the second involved canonical representation techniques.

The automatically generated candidates were constructed under the following constraints. Each candidate must (1) be a lattice equation in terms of meet and join, (2) not necessarily hold for all lattices, (3) hold for all Boolean algebras, and (4) hold for the least nonmodular lattice N_5. Several thousand identities were generated, and about 80 were proved to be Huntington identities. Some of the proofs (e.g., H58 of Theorem 3) were easy for the theorem provers, and some were difficult, requiring some human guidance. The Huntington identities were then classified according to the nonmodular lattices in which they hold, as in Table 2. Those that could be proved to be equivalent (mod lattice theory and duality) to others were removed, and in some cases, if implications (mod lattice theory and duality) could be proved, the stronger ones were removed. The result is the set listed in Table 2.

Classical lattice theory is abundant with computational techniques for finding nonmodular lattice identities. Most of these originate from the seminal paper of Ralph Mckenzie [9]. For example, he gives equational bases for various covers of $\mathcal{V}(N_5)$. These bases alone lead to several nonmodular Huntington laws. For example, (H69) discussed in the beginning of the paper is simply a weaker three-variable version of McKenzie's four-variable identity (η_2) [9, p. 7]. Naturally, this weaker version holds in several covers of N_5. Another technique is to start from, say, the basic distributive law $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and consider the new lattice identity $f = (x \wedge y) \vee (x \wedge z)$, where the lattice term f is obtained from $x \wedge (y \vee z)$ by replacing y with $y \wedge (x \vee z)$ and z with $z \wedge (x \vee y)$. This transform may be construed as N_5-correction of distributivity. This results in

$$x \wedge ((y \wedge (x \vee z)) \vee (z \wedge (x \vee y))) = (x \wedge y) \vee (x \wedge z)$$

which is precisely our (H82). While this weak form does not imply distributivity, it does retain some flavor of distributivity: namely, it implies CD-$\vee$ and hence defines a Huntington variety. Another rich source is that of Jónsson and Rival [5] who gave a family of lattice laws that imply either SD-$\vee$ or SD-$\wedge$. We selected several of those and experimented with theorem-proving programs to obtain weaker Huntington laws. Note that all of these identities are, of course, nonmodular because all of them are, by design, valid in N_5.

Appendix

Proof of Theorem 2, Part CD-$\vee$. This proof was produced by the program Prover9 [8]. The input and output files can be found on the supporting Web page [12].

Notes on the Prover9 and Otter proofs.

(1) Proofs are by contradiction.
(2) Terms x, y, z, u, v, w are variables, and A, B, C, D are constants.
(3) Implications are written as disjunctions.

(4) The justification "$m \to n$" means that an instance of clause m is used to replace a term in an instance of clause n, and "$; i, j, \cdots$" means simplification with $i, j, \cdots$.

13	$x \vee y = y \vee x$	[input]
14	$x \wedge y = y \wedge x$	[input]
15	$(x \vee y) \vee z = x \vee (y \vee z)$	[input]
16	$(x \wedge y) \wedge z = x \wedge (y \wedge z)$	[input]
17	$x \wedge (x \vee y) = x$	[input]
18	$x \vee (x \wedge y) = x$	[input]
19	$x \vee x' = 1$	[input]
20	$x \wedge x' = 0$	[input]
21	$x \vee y \neq 1 \ \mid \ x \wedge y \neq 0 \ \mid \ x' = y$	[input]
22	$A \wedge B = A$	[input]
23	$A' \vee B' \neq A'$	[input]
24	$x \vee y \neq x \vee z \ \mid \ x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$	[input]
26	$x \wedge (y \wedge z) = y \wedge (x \wedge z)$	[14 $\to$ 16; 16]
32	$x \vee ((x \wedge y) \vee z) = x \vee z$	[18 $\to$ 15]
37	$x \vee (x' \vee y) = 1 \vee y$	[19 $\to$ 15]
39	$x \wedge 1 = x$	[19 $\to$ 17]
42	$x \vee 0 = x$	[20 $\to$ 18]
43	$A \wedge (B \wedge x) = A \wedge x$	[22 $\to$ 16]
51	$x \vee y \neq 1 \ \mid \ x \wedge (y \vee x') = 0 \vee (x \wedge y)$	[19 $\to$ 24; 20 13]
60	$1 \wedge x = x$	[39 $\to$ 14]
66	$0 \vee x = x$	[42 $\to$ 13]
69	$x \vee y \neq 1 \ \mid \ x \wedge (y \vee x') = x \wedge y$	[51; 66]
72	$1 \vee x = 1$	[60 $\to$ 17]
73	$x \vee (x' \vee y) = 1$	[37; 72]
75	$0 \wedge x = 0$	[66 $\to$ 17]
79	$x \wedge (y \wedge x') = y \wedge 0$	[20 $\to$ 26]
81	$x \vee 1 = 1$	[72 $\to$ 13]
83	$x \wedge 0 = 0$	[75 $\to$ 14]
84	$x \wedge (y \wedge x') = 0$	[79; 83]
103	$A \wedge (A' \vee B') \neq 0$	[21 73 23]
170	$x \vee (x \wedge y)' = 1$	[19 $\to$ 32; 81]
185	$x \vee (y \wedge x)' = 1$	[14 $\to$ 170]
194	$B \vee A' = 1$	[22 $\to$ 185]
891	$B \wedge (A' \vee B') = B \wedge A'$	[69 194]
5852	$A \wedge (A' \vee B') = 0$	[891 $\to$ 43; 84]
5853	$\square$	[5852 103]

Proof of Theorem 3. This proof was produced by the program Prover9 [8]. The input and output files can be found on the supporting Web page [12].

29	$x \vee y = y \vee x$	[input]
37	$x \vee (y \wedge x) = x$	[input]
38	$x \vee (x \vee y) = x \vee y$	[input]
40	$x \wedge x' = 0$	[input]
47	$x \vee 0 = x$	[input]

48	$0 \vee x = x$	[input]
49	$x \vee y \neq 1 \ \mid\ x \wedge y \neq 0 \ \mid\ x' = y$	[input]
50	$x \wedge (x \vee y)' = 0$	[input]
51	$x \vee (y \vee (x \vee y)') = 1$	[input]
52	$x \wedge (y \vee ((x \vee y) \wedge (z \vee (x \wedge y)))) = x \wedge (y \vee z)$	[input]
53	$A \wedge B = A$	[input]
54	$A' \vee B' \neq A'$	[input]
103	$A \vee B = B$	[53 → 37; 29]
107	$A \wedge B' = 0$	[103 → 50]
109	$A \wedge (B' \vee ((A \vee B') \wedge x)) = A \wedge (B' \vee x)$	[107 → 52; 47]
2392	$A \wedge (B' \vee (A \vee B')') = 0$	[40 → 109; 29 48 107]
2439	$B' \vee (A \vee B')' = A'$	[49 51 2392]
2481	$A' \vee B' = A'$	[2439 → 38; 29 2439]
2482	$\square$	[2481 54]

References

[1] M. E. Adams and J. Sichler. Lattice with unique complementation. *Pac. J. Math.*, 92:1–13, 1981.

[2] R. P. Dilworth. Lattices with unique complements. *Trans. AMS*, 57:123–154, 1945.

[3] G. Grätzer. *General Lattice Theory*. Birkhauser Verlag, 1975.

[4] E. V. Huntington. Sets of independent postulates for the algebra of logic. *Trans. AMS*, 5:288–309, 1904.

[5] B Jónsson and I. Rival. Lattice varieties covering the smallest nonmodular variety. *Pacific J. Math.*, 82(2):463–478, 1979.

[6] W. McCune. Otter 3.0 Reference Manual and Guide. Tech. Report ANL-94/6, Argonne National Laboratory, Argonne, IL, 1994. Also see `http://www.mcs.anl.gov/AR/otter/`.

[7] W. McCune. Mace4 Reference Manual and Guide. Tech. Memo ANL/MCS-TM-264, Mathematics and Computer Science Division, Argonne National Laboratory, Argonne, IL, August 2003.

[8] W. McCune. Prover9. `http://www.mcs.anl.gov/~mccune/prover9/`, 2005.

[9] R. McKenzie. Equational bases and nonmodular lattice varieties. *Trans. AMS*, 174:1–43, 1972.

[10] R. Padmanabhan. Huntington's conjecture revisited. To Appear.

[11] R. Padmanabhan. A first-order proof of a theorem of Frink. *Algebra Universalis*, 13(3):397–400, 1981.

[12] R. Padmanabhan, W. McCune, and R. Veroff. Lattice laws forcing distributivity under unique complementation: Web support. `http://www.mcs.anl.gov/~mccune/papers/uc-lattice/`, 2005.

[13] V. N. Salii. *Lattices with Unique Complements*. American Mathematical Society, Providence, RI, 1988.

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