Semantic Guidance

(and other methods)

for Saturation Provers

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An Equational Problem

$$f(f(y,x), f(f(f(x,x),z), f(f(f(f(f(x,y),z),z),x), f(x,u)))) = x$$

$$\overset{?}{\Rightarrow}$$

$$f(x, f(y, f(x, f(z,z)))) = f(x, f(z, f(x, f(y,y))))$$

Saturation Provers for First-order and Equational Logic

- Inference
  - Proof by refutation
  - Make inferences (under some restrictions), keeping formulas simplified
- Completeness
  - If the conjecture is a theorem, a proof will eventually be found
- Strategy
  - Term ordering for inference and simplification
  - Selection function for application of inference rules
  - Special-purpose strategies (may be incomplete)
- Examples: Otter, E, Vampire, Waldmeister, Prover9

The Inference Loop

- Maintain two sets of formulas
  - *active*: available for making inferences
  - *passive*: waiting to participate in the search
- Loop:
  - Select a formula C from *passive* (the *given clause*);
  - move C from passive to *active*;
  - make inferences from C and other formulas in *active*;
  - simplify inferred formulas and append to *passive*;
- Two versions: *Otter loop* and *Discount loop*
  - differ on when and how simplification occurs

Theory vs. Practice

Completeness Results

- Even though an inference is not needed,
  - it might be very helpful in practice.
- Even though an inference may be necessary in general,
  - it might be wasteful in practice.

Restriction vs. Guidance

- Restriction: Constrain application of inference rules
- Guidance: Selection of the *given clause*

Three Promising Strategies

1. Semantic Guidance

   – using interpretations to help select the *given clause*

2. Variable Ordering

   – extending the term ordering with variables

   – $(y * x) \succ (x * y)$

   – incomplete

3. Introduce Equational Definitions

   – emphasize “important” terms, e.g.,

$$g(x, y) = f(f(x, x), f(y, y))$$

Programs for Experimenting with Strategies

- Prover9
  - theorem prover for first-order and equational logic
  - Otter's (and EQP's) successor
- Mace4
  - searches for finite models and counterexamples

Semantic Guidance

Using interpretations to help select the *given clause*

Semantic Inference Rules for First-order and Equational Deduction

- Based on *interpretations* (explicit or implicit, finite or infinite)
- Semantic Resolution (Slagle 1965)
  - given an interpretation I, for each (binary) inference, at least one parent must be false in I.
  - example: *positive resolution* uses the (implicit) interpretation in which all atomic formulas are false.
- *Set of support resolution* (Wos, 1964): justified by any interpretation of the axioms, and the semantic restriction applies only to input clauses.

Semantic Inference Rules (continued)

- Semantic Paramodulation
- Semantic Resolution for Horn Sets (Henschen 1976)
  - If one of the parents is a positive unit false in I, the child must also be false in I.
  - applies also to paramodulation

Semantic Inference with Explicit Interpretations

- not much work, not much success
  - incompatibilities with inference rules based on term orderings and simplification
- Powerful finite-model finding programs, starting about 1990
  - Finder, MGTP, Mace2, Falcon, SEM, Mace4, Paradox.
- The SCOTT series of provers (Slaney, Hodgson, et al).
  - dynamic interpretations, partial interpretations, multiple interpretations
  - guidance and well as restriction

Semantic Guidance

- Assume the *Otter loop* or the *Discount loop*.
- Selecting the next clause for making inferences (the *given clause*).
  - weighting function: lightest first
  - age: breadth first
  - age/weight ratio
- For semantic guidance, specify a 3-way ratio:
  - age,
  - lightest clause *false* in I,
  - lightest clause *true* in I.

Lattice Theory Example: The Problem

In lattice theory (LT), what is the relationship between the following two identities?

$$x \wedge (y \vee (x \wedge z)) = x \wedge (y \vee (z \wedge ((x \wedge (y \vee z)) \vee (y \wedge z)))) \qquad \text{(H2)}$$

$$(x \wedge y) \vee (x \wedge z) = x \wedge ((y \wedge (x \vee z)) \vee (z \wedge (x \vee y))) \qquad \text{(H82)}$$

- LT, H2 $\rightarrow$ H82 ? *No*. Mace4 easily finds a counterexample of size 6.
- LT, H82 $\rightarrow$ H2 ?

Semantic guidance for LT, H82 $\rightarrow$ H2

- The conjecture has the form *theory, hypotheses* $\rightarrow$ *conclusion*.
- Consider a lattice (the theory) in which the H2 (the conclusion) is false.

Lattice Theory Example: Guiding Interpretation

Mace4 Input

```
formulas(assumptions).

  % Lattice theory

  x v y = y v x.
  (x v y) v z = x v (y v z).
  x ^ y = y ^ x.
  (x ^ y) ^ z = x^ (y ^ z).
  x ^ (x v y) = x.
  x v (x ^ y) = x.

end_of_list.

formulas(goals).  % falsify these formulas

  x ^ (y v (x ^ z)) = x ^ (y v (z ^ ((x ^ (y v z)) v (y ^ z)))) # label(H2).

end_of_list.
```

Lattice Theory: Guiding Interpretation

Mace4 Result

```
interpretation( 6, [number=1, seconds=0], [

        function(c1, [2]),
        function(c2, [3]),
        function(c3, [4]),

        function(^(_,_), [ 0, 0, 0, 0, 0, 0,
                           0, 1, 2, 3, 4, 5,
                           0, 2, 2, 0, 0, 0,
                           0, 3, 0, 3, 5, 5,
                           0, 4, 0, 5, 4, 5,
                           0, 5, 0, 5, 5, 5]),

        function(v(_,_), [ 0, 1, 2, 3, 4, 5,
                           1, 1, 1, 1, 1, 1,
                           2, 1, 2, 1, 1, 1,
                           3, 1, 1, 3, 1, 3,
                           4, 1, 1, 1, 4, 4,
                           5, 1, 1, 3, 4, 5])]).
```

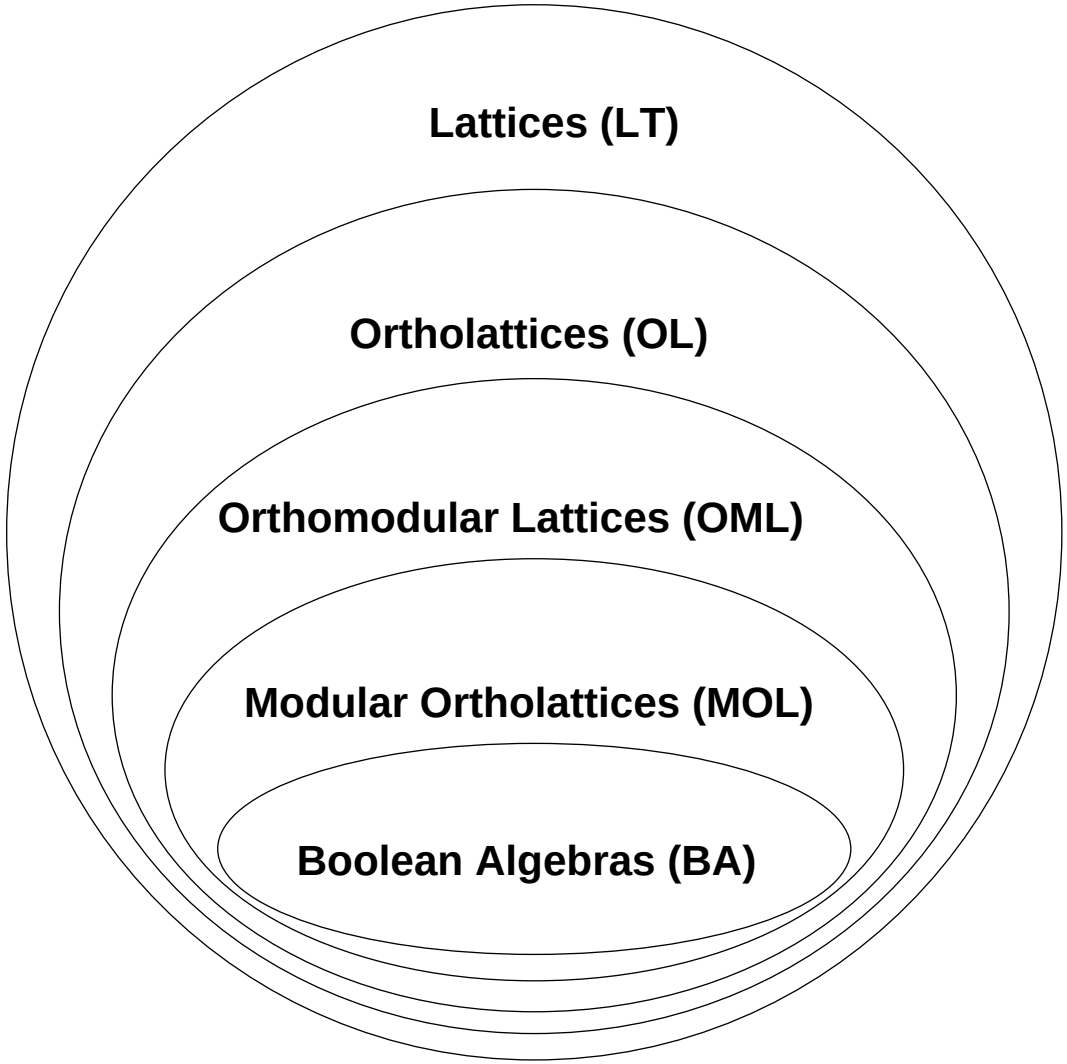
Lattice Theory: Search with Guiding Interpretation

- Give Prover9 the guiding interpretation.
- Tell Prover9 to use the selection ratio (age=1 : false=4 : true=1).
- Proof in 10 seconds.
- Proof in 1 hour without guidance.

How does the interpretation help to find a proof?

A Chain of Varieties: LT, OL, OML, MOL, BA

As structures (algebras):



A Chain of Varieties: LT, OL, OML, MOL, BA

Lattice Theory (LT)

$$\begin{array}{ll} (x \vee y) \vee z = x \vee (y \vee z). & (x \wedge y) \wedge z = x \wedge (y \wedge z). \\ x \vee y = y \vee x. & x \wedge y = y \wedge x. \\ x \vee (x \wedge y) = x. & x \wedge (x \vee y) = x. \end{array}$$

For Ortholattices (OL), include a unary operation with the following constraints.

$$\begin{array}{l} x \wedge y = (x' \vee y')'. \\ x'' = x. \\ x \vee x' = 1. \end{array}$$

For Orthomodular Lattices (OML), add the following constraint.

$$x \vee (x' \wedge (x \vee y)) = x \vee y.$$

For Modular Ortholattices (MOL), add the following constraint.

$$x \vee (y \wedge (x \vee z)) = x \vee (z \wedge (x \vee y)).$$

For Boolean Algebras (BA), add the following constraint.

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

Modular Ortholattice (MOL) Single Axiom: The Problems

Consider the equations

$$f(f(y,x),f(f(f(x,x),z),f(f(f(f(f(x,y),z),z),x),f(x,u)))) = x \qquad \text{(MOL)}$$

$$f(x,f(y,f(x,f(z,z)))) = f(x,f(z,f(x,f(y,y)))) \qquad \text{(M)}$$

$$f(x,f(f(y,z),f(y,z))) = f(y,f(f(x,z),f(x,z))) \qquad \text{(A)}$$

- (MOL) is a single axiom for modular ortholattices in terms of the Sheffer stroke (NAND).
- Problem 1 (modularity property): (MOL) $\rightarrow$ (M).
- Problem 2 (associativity property): (MOL) $\rightarrow$ (A).

MOL Single Axiom: Selecting Interpretations

- Modularity property.
  - Consider the chain of varieties: OL, OML, MOL, BA

    (Ortholattices, Orthomodular Lattices, Modular Ortholattices, Boolean Algebra).
  - Let OML be the “theory”, and use the smallest non-modular OML (size 10) to guide the search.
  - Proof in 10 minutes with guidance; proof in 50 minutes without.
- Associativity property.
  - How do we find a nonassociative “theory” in which to work?
  - Simple answer: Consider nonassociative interpretations that satisfy other MOL properties. Pick one of size 8.
  - Proof in 44 seconds with guidance; proof in 162 seconds without.

Variable Ordering

Extending the term ordering with variables

Variable Ordering: Background

- Proof search based on term ordering and term rewriting.
- A simplification ordering allows equations to be used as rewrite rules.
  - E.g., KBO (Knuth-Bendix Ordering), LPO (Lexicographic Path Ordering).
- Ordinarily, term rewriting always produces smaller terms.
  - Orientable rules, e.g, $x'' = x$, are always applied.
  - A non-orientable rule, e.g., $x * y = y * x$, is applied only if it produces a smaller term.
- Many proof procedures based on these term orderings are *complete*.

Variable Ordering: An Incomplete Strategy

- When deciding whether to apply a non-orientable rewrite rule, include variables in the term ordering:

  variables $\prec$ constants

  $x \prec y \prec z \prec u \prec v \prec \cdots$

  In other words, treat variables as constants when comparing terms.

- We have no theoretical basis for this method.
- The method is incomplete (it can block all proofs).
- It can be used with KBO or with LPO.

Variable Ordering: Median Algebra Problem

Axioms for Median Algebra

$$
\begin{aligned}
&f(x,y,z) = f(z,x,y)\\
&f(x,y,z) = f(x,z,y)\\
&f(x,x,y) = x\\
&f(f(x,w,y),w,z) = f(x,w,f(y,w,z))
\end{aligned}
$$

Conclusion

$$f(f(x,y,z),u,w) = f(x,f(y,u,w),f(z,u,w))$$

- Note that the first two axioms allow the arguments to be permuted in any way.

Variable Ordering: Canonicalization

- Consider the non-orientable rewrite rules

$$f(x, y, z) = f(z, x, y)$$
$$f(x, y, z) = f(x, z, y)$$

- If $v_1 \prec v_2 \prec v_3 \prec v_4 \prec v_5$,
  the canonical form of

$$f(v_4, f(v_3, v_5, v_1), v_2)$$

  is

$$f(v_2, v_4, f(v_1, v_3, v_5))$$

- This (incomplete) method drastically reduces the search space.

- Proof in about 15 minutes (no proof without variable ordering).

Introducing Equational Definitions

Introducing Equational Definitions

Emphasize “important” terms.

- A very simple, human-like strategy.
  - Input equational definitions, e.g., define a binary operation g:

$$g(x, y) = f(f(x, x), f(y, y))$$

  - Arrange the term ordering so that the defined operation is introduced whenever possible, that is, use the definition as a rewrite rule:

$$f(f(x, x), f(y, y)) \longrightarrow g(x, y)$$

Definition: Ortholattice Example

- The Problem (f is the Sheffer stroke, i.e., NAND):

  Given: $f(f(f(f(x,y),f(y,z)),u),f(y,f(f(y,f(f(x,x),x)),z))) = y$
  Prove: $f(x,f(f(y,z),f(y,z))) = f(y,f(f(x,z),f(x,z)))$

- Definitions:

  Join: $x \vee y = f(f(x,x),f(y,y))$
  Meet: $x \wedge y = f(f(x,y),f(x,y))$
  Complementation: $x' = f(x,x)$

- Much easier to prove with definitions

  – proof is in terms of the defined operations

  – shorter equations in proof

The Robbins Problem with Definitions(1)

- Given

$$x \vee y = y \vee x \qquad \text{(Commutativity)}$$
$$(x \vee y) \vee z = x \vee (y \vee z) \qquad \text{(Associativity)}$$
$$((x \vee y)' \vee (x \vee y')')' = x \qquad \text{(Robbins)}$$

- Prove

$$(x \vee y')' \vee (x' \vee y')' = y \qquad \text{(Huntington)}$$

- First proved by EQP in 1996.
- No proof (human or computer) since then?

The Robbins Problem with Definitions (2)

- No proof found without definitions (search for several weeks)
- Consider the definition.

$$f(x) = x + (x + x')'$$

  Proof in about half an hour.

- Consider the pair of definitions.

$$g(x) = (x + x')'$$
$$h(x) = x + (x + (x + g(x)))$$

  Proof in about 45 seconds.

Concluding Remarks

- Selecting interpretations and definitions.
  - input form expert (mathematician)
  - automatically, by some rules
- Incomplete strategies, e.g., variable ordering.
- Work is needed in the following areas.
  - experimentation on hard problems
  - analysis of hard proofs
  - application to non-equational problems
  - work with mathematicians on problems of interest to them

Thank You