Conditions for Exact Resultants using the Dixon Formulation

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1 Introduction

A structural criteria on multivariate polynomial systems is developed such that the generalized Dixon formulation of multivariate resultants [6, 12] as well as the associated sparse resultant construction recently obtained by the authors from this formulation [5] computes the resultant exactly, i.e. these constructions do not produce any extraneous factors in the resultant computations for such polynomial systems. This result is of considerable significance since extraneous factors arising in resultant computations of polynomial systems is a key problem faced when any multivariate resultant method for simultaneously eliminating many variables is used for elimination in a variety of applications including computer vision, robotics and kinematics, control theory, solid and geometric modeling, geometry theorem proving, biology, etc. A long-term goal of this research is to be able to identify subsets of monomials in relation to the supports of polynomials in a polynomial system that can lead to extraneous factors in the computation of a *projection operator* (which is a nonzero multiple of the resultant) from the multivariate generalized Dixon resultant formulation.

Towards this objective, the notion of a *Dixon-exact* polytope is introduced; it is proved that the Dixon resultant formulation produces the exact resultant for generic unmixed systems whose support is a Dixon-exact polytope. A geometric operation, called *direct-sum*, on polytopes is defined that preserves the property of polytopes being Dixon-exact. Generic n-degree systems for which the Dixon formulation is known to compute exact resultants [6, 13] are shown to be a special case of generic unmixed polynomial systems whose support is Dixon-exact. Multigraded systems introduced by Strumfels and Zelvinsky [15] for which they gave a Sylvester type formula for resultants are also shown to be a special case of generic unmixed polynomial systems whose support is Dixon-exact. In other words, both the generalized Dixon formulation as well as the sparse Dixon resultant matrices constructed from them produce exact resultants for n-degree as well multigraded polynomial systems. Besides these systems, other unmixed polynomial systems with Dixon-exact support are identified for which exact resultants can be computed, thus extending results in [6, 15, 13].

Using the techniques discussed in [11], a wide class of polynomial systems can be identified for which the Dixon formulation produces exact resultants. This study was motivated by our attempts to understand why and how the generalized Dixon method is successful in computing the exact resultant of the famous Stewart platform problem using for a particular variable ordering, as discussed in [11].

In [11], it was proved that for a class of polynomial systems, the projection operator computed by the generalized Dixon formulation can be related to the projection operator computed by the same method for a simpler polynomial system (with polynomials whose support is a smaller Newton polytope and lower degrees). This result can be exploited in many different ways. Firstly, the projection operator for such smaller polynomial systems can be computed much faster using less memory space, than similar computations for the larger systems. Secondly, extraneous factors in the projection operator can be shown to be powers of the extraneous factors in the projection operator of the related smaller polynomial system. In other words, if the generalized Dixon formulation does not generate extraneous factors in the projection operator of the smaller system, then it is guaranteed not to do so for the larger one either. It therefore becomes useful to identify polynomial systems for which the extended Dixon formulation computes the exact resultant.

For the bivariate case, we provide a complete analysis of monomials in a polynomial system vis a vis their role in producing extraneous factors in a projection operator computing using the generalized Dixon formulation. Such an analysis is likely to give insights for the general case of elimination of arbitrarily many variables.

The problem of extraneous factors is not peculiar to the elimination method based on the generalized Dixon formulation. In fact, none of the resultant based elimination methods – the Macaulay formulation, Dixon formulation or the sparse resultant formulation, compute the exact resultant of arbitrary nonhomogeneous polynomial systems. Instead, these methods compute various multiples of the resultant, known as projection operators, which may contain extraneous factors besides the resultant. Since the information about the solutions of a polynomial system is completely contained in its resultant, the extraneous factors in a projection operator do not offer any additional information. Instead, they make it more difficult to identify the resultant in a projection operator, as for each factor in a projection operator, it must be checked whether it is extraneous or if it is a part of the resultant, and this check can be resource-consuming. The presence of extraneous factors can also make computing a projection operator impractical, since extraneous factors can increase the total degree of the projection operator considerably, and the computational complexity of the projection operator using the generalized

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Dixon method (as well as other resultant formulations) is determined by its degree.

The results reported in this paper extend our earlier results about the generalized Dixon formulation, a method for simultaneously eliminating many variables for a large class of polynomial systems and computing a projection operator from which the resultant can be extracted [12]. This method has been experimentally found to be superior in performance on a wide variety of examples, in comparison with other elimination methods including Macaulay resultants, sparse resultants [3, 14, 8], the characteristic set construction [4], and the Gröbnerbasis construction [1, 2]. The method takes less time, less space, as well as the extraneous factors seem to be fewer (except in the case of the Gröbnerbasis method which gives the exact resultant) [9]. We also proved that for the unmixed case, the Dixon formulation, in fact, *implicitly* exploits the sparse structure of the polynomial system, i.e., its computational complexity is governed by the Newton polytope of the unmixed system, not by the Bezout bound as is the case for Macaulay resultants [10]. Recently, we have been able to develop a simple construction for efficiently generating sparse resultant matrices using the Dixon formulation, in contrast to other known techniques for generating sparse resultant matrices based on explicitly exploiting the support of the polynomial system.

2 Definitions & Notation

Definition 2.1 *Given a polynomial $f = c_{\alpha_1}\mathbf{x}^{\alpha_1} + \cdots + c_{\alpha_n}\mathbf{x}^{\alpha_n} \in \mathbb{C}[x_1,\ldots,x_d]$ and $\mathbf{x}^{\alpha_i} = x_1^{\alpha_{i,1}} x_2^{\alpha_{i,2}} \ldots x_d^{\alpha_{i,d}}$, let the support of f be the set $\mathcal{A} = \{\alpha_1,\ldots,\alpha_n\}$, where $\alpha_i \in \mathbb{N}^d$ and $c_{\alpha_i} \neq 0$. Define a* ***support map***

$$\begin{aligned} \mathsf{S}_{\mathbf{x}} : \mathbb{C}[x_1,\ldots,x_d] &\rightarrow \mathbb{N}^d, \\ \mathsf{S}_{\mathbf{x}}(f) &= \mathcal{A}. \end{aligned}$$

Definition 2.2 *Define $\pi_i(\mathbf{x}^\alpha) = \overline{x}_1^{\alpha_1}\cdots\overline{x}_i^{\alpha_i} x_{i+1}^{\alpha_{i+1}}\cdots x_d^{\alpha_d}$, where $i \in \{0,\ldots,d\}$ and $\overline{x}_i$ are new variables. Note that $\pi_0(\mathbf{x}^\alpha) = \mathbf{x}^\alpha$. π_i can be extended in a natural way to polynomials as arguments:*

$$\pi_i(f(x_1,\ldots,x_d)) = f(\overline{x}_1,\ldots,\overline{x}_i,x_{i+1},\ldots,x_d).$$

Definition 2.3 *Given $P = \{f_0,f_1,\ldots,f_d\} \subset \mathbb{C}[x_1,\ldots,x_d]$, define its* ***Dixon polynomial*** *as*

$$\theta(f_0,\ldots,f_d) = \prod_{i=1}^{d} \frac{1}{\overline{x}_i - x_i} \begin{vmatrix} \pi_0(f_0) & \pi_0(f_1) & \cdots & \pi_0(f_d) \\ \pi_1(f_0) & \pi_1(f_1) & \cdots & \pi_1(f_d) \\ \vdots & \vdots & \ddots & \vdots \\ \pi_d(f_0) & \pi_d(f_1) & \cdots & \pi_d(f_d) \end{vmatrix}.$$

Hence $\theta(f_0,f_1,\ldots,f_d) \in \mathbb{C}[x_1,\ldots,x_d,\overline{x}_1,\ldots,\overline{x}_d]$, where $\overline{x}_1,\overline{x}_2,\ldots,\overline{x}_d$ are new variables. Note that the support of the Dixon polynomial depends on the variable order used in constructing it.

A polynomial system $P = \{f_0,\ldots,f_d\}$ is called *unmixed* if each polynomial has the same support, i.e., for all i, $\mathsf{S}_{\mathbf{x}}(f_i) = \mathcal{A}$. For unmixed generic systems, we will use $\theta_{\mathcal{A}} = \theta(f_0,f_1,\ldots,f_d)$, to stress its dependence on the support of the polynomial system.

Definition 2.4 *A Dixon polynomial $\theta(f_0,\ldots,f_d)$ can be written in bilinear form as*

$$\theta(f_0,f_1,\ldots,f_d) = \overline{X}\Theta X,$$

where $\overline{X} = \{\overline{x}^\alpha | \overline{x}^\alpha \in \theta\}$ and similarly, $X = \{x^\alpha | x^\alpha \in \theta\}$. The matrix Θ is called the ***Dixon Matrix****.*

As proved in [10, 13], the determinant of a maximal minor of the Dixon matrix Θ is, in general, a *projection operator*, a non-trivial multiple of the resultant, provided that the Dixon matrix Θ has an independent column. Each entry in Θ is a polynomial in the coefficients of the polynomials in P, and its degree in the coefficients of any single polynomial is at most 1. Hence, the projection operator computed using the Dixon formulation can be at most of degree $|X|$ in the coefficients of any single polynomial. Since the degree of a resultant of a generic unmixed system can be predicted in advance, $|X|$ reveals if any extraneous factor exists in the determinant of a maximal minor, and if so, gives a lower bound on the degree of the extraneous factor.

Since $|X| = |\mathsf{S}_{\mathbf{x}}(\theta)|$, we are interested in estimating $|\mathsf{S}_{\mathbf{x}}(\theta)|$. The support of the Dixon polynomial can also be analyzed in terms of $\overline{x}_i$ variables, but this is equivalent to doing the analysis in terms of x_i variables if the variable order is completely reversed.

Definition 2.5 *Given a polynomial $f \in \mathbb{C}[x_1,\ldots,x_d]$, $\mathsf{Vol}_d(\mathcal{A})$ denotes the Euclidean volume of the convex hull of $\mathcal{A} = \mathsf{S}_{\mathbf{x}}(\theta)$.*

If $\mathcal{A}$ has dimension smaller that d, then $\mathsf{Vol}_d(\mathcal{A}) = 0$.

It is known that the degree of the toric resultant of a generic unmixed polynomial system with support $\mathcal{A}$ in the coefficients of any polynomial is $d!\mathsf{Vol}_d(\mathcal{A})$.

The Dixon resultant formulation yields, in general, a non-trivial multiple of the affine resultant, and it generally has higher degree than its toric counterpart. Here, we are mostly concerned with supports which contain the origin, i.e., polynomial systems whose polynomials contain a constant term. In this case, toric and affine resultants coincide.

Proposition 2.1 *Given an unmixed generic polynomial system $P = \{f_0,f_1,\ldots,f_d\}$ with a support $\mathcal{A}$,*

$$|\mathsf{S}_{\mathbf{x}}(\theta_{\mathcal{A}})| \geq d!\mathsf{Vol}_d(\mathcal{A}).$$

Proof: If the resultant of P exists, then there exist a maximal minor in its Dixon matrix whose determinant is a non-zero projection operator (see [10], [7]). Since the degree of the resultant is $d!\mathsf{Vol}_d(\mathcal{A})$ in the coefficients of any polynomial in P, the size of the Dixon matrix has to be at least that big, and hence, the size of the support of the Dixon polynomial. □

The Dixon polynomial of a polynomial system P can be decomposed into a sum of smaller Dixon polynomials of polynomial systems with $d+1$ monomials. This nice identity for the Dixon polynomial is used later in proofs.

Proposition 2.2 *Given a polynomial system $P = \{f_0,f_1,\ldots,f_d\}$, let $\mathcal{A} = \cup_{i=0}^{d}\mathsf{S}_{\mathbf{x}}(f_i)$. For $\sigma \subseteq \mathcal{A}$, $|\sigma| = d+1$, let $\sigma(\mathbf{c})$ stand for the matrix whose $(i,j)^{th}$ entry is c_{i,σ_j}, where σ_j is the j-th element of σ, viewed as a sequence, c_{i,σ_j} is the coefficient of monomial $\mathbf{x}^{\sigma_j}$ in f_i; similarly, let $\sigma(\mathbf{x})$ be the matrix whose $(i,j)^{th}$ entry is $\pi_i(\mathbf{x}^{\sigma_j})$. Then*

$$\theta(f_0,f_1,\ldots,f_d) = \prod_{i=1}^{d} \frac{1}{\overline{x}_i - x_i} \sum_{\substack{\sigma \subseteq \mathcal{A} \\ |\sigma| = d+1}} |\sigma(\mathbf{c})||\sigma(\mathbf{x})|.$$

Proof: Let $f_i = \sum_{j=1}^{n} c_{i,\alpha_j}\mathbf{x}^{\alpha_j}$, (assuming 0 for coefficients if necessary). The Dixon polynomial of P is given by

$$\theta(f_0,\ldots,f_d) =$$

$$= \prod_{i=1}^{d}\frac{1}{\overline{x}_i - x_i} \left| \begin{pmatrix} \pi_0(f_0) & \pi_0(f_1) & \cdots & \pi_0(f_d) \\ \pi_1(f_0) & \pi_1(f_1) & \cdots & \pi_1(f_d) \\ \vdots & \vdots & \ddots & \vdots \\ \pi_d(f_0) & \pi_d(f_1) & \cdots & \pi_d(f_d) \end{pmatrix}^T \right|$$

$$= \prod_{i=1}^{d}\frac{1}{\overline{x}_i - x_i} \left| \begin{pmatrix} c_{0,\alpha_1} & c_{0,\alpha_2} & \cdots & c_{0,\alpha_n} \\ c_{1,\alpha_1} & c_{1,\alpha_2} & \cdots & c_{1,\alpha_n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{d,\alpha_1} & c_{d,\alpha_2} & \cdots & c_{d,\alpha_n} \end{pmatrix} \begin{pmatrix} \pi_0(\mathbf{x}^{\alpha_1}) & \pi_1(\mathbf{x}^{\alpha_1}) & \cdots & \pi_d(\mathbf{x}^{\alpha_1}) \\ \pi_0(\mathbf{x}^{\alpha_2}) & \pi_1(\mathbf{x}^{\alpha_2}) & \cdots & \pi_d(\mathbf{x}^{\alpha_2}) \\ \vdots & \vdots & \ddots & \vdots \\ \pi_0(\mathbf{x}^{\alpha_n}) & \pi_1(\mathbf{x}^{\alpha_n}) & \cdots & \pi_d(\mathbf{x}^{\alpha_n}) \end{pmatrix} \right|$$

$$= \prod_{i=1}^{d}\frac{1}{\overline{x}_i - x_i} \sum_{\substack{\sigma\subseteq\mathcal{A} \\ |\sigma|=d+1}} |\sigma(\mathbf{c})||\sigma(\mathbf{x})|$$

(by Cauchy-Binet Formula).

The above identity shows that if generic coefficients are assumed in the polynomial system, then the support of the Dixon polynomial depends entirely on the support of the polynomial system.

In [5], we have given a construction for sparse resultant matrices from the Dixon formulation. It is proved that whenever the Dixon formulation computes the exact toric resultant, the sparse resultant matrix obtained from the Dixon formulation also computes the exact toric resultant. So the results below also apply to sparse resultant matrices constructed from the Dixon formulation as discussed in [5].

3 Dixon-Exact Support, Basis Simplex

Definition 3.1 *Given a generic unmixed polynomial system P with support $\mathcal{A}$, $\mathcal{A}$ is called Dixon-exact if there exists a variable order resulting in the Dixon polynomial such that*

$$|\mathsf{S}_{\mathbf{x}}(\theta_{\mathcal{A}})| = d!\mathsf{Vol}_d(\mathcal{A}).$$

Definition 3.2 *A point $p \in \mathcal{A}$ is called Dixon-interior if*

$$\mathsf{S}_{\mathbf{x}}(\theta_{\mathcal{A}}) = \mathsf{S}_{\mathbf{x}}(\theta_{\mathcal{A}-\{p\}}).$$

In the literature, a point is called interior with respect to a support $\mathcal{A}$ if it is not in $\mathcal{A}$, and it belongs to the convex hull of $\mathcal{A}$. It is known that the presence of monomials corresponding to points not in the support of a polynomial system but inside the convex hull do not change the degree of its resultant. Yet there exist examples where given an unmixed polynomial system with support $\mathcal{A}$, and point p not in $\mathcal{A}$ but in its convex hull, the Dixon matrix of $\mathcal{A}\cup\{p\}$ is bigger than that of $\mathcal{A}$, resulting in the projection operator containing extraneous factors. This is the rationale for introducing the above definition which is much more restricted.

The above observation is also true for sparse matrices. For example, in the bivariate case, a polynomial system with support $\{(0,0),(2,0),(0,2)\}$ results in the Dixon matrix of size 4×4, and the sparse matrix of size 12×12. If a monomial corresponding to the point $(1,1)$ not in the support but in its convex hull is added to the polynomial system, the Dixon matrix becomes of size 5×5, and the smallest sparse resultant matrix is of size 14×14, whereas the degree of the resultant remains 4 in the coefficients of any polynomial.

Definition 3.3 *Let P and Q be supports.*

- *The* ***Minskowski Sum*** *of $\mathcal{P}$ and $\mathcal{Q}$, denoted by $\mathcal{P}+\mathcal{Q}$, is*
$$\mathcal{P}+\mathcal{Q} = \{p+q | p\in\mathcal{P} \text{ and } q\in\mathcal{Q}\},$$
where $p+q$ is the regular vector sum.
We also use the notation $p+\mathcal{Q}$ to stand for $\{p\}+\mathcal{Q}$.
- *For any non-negative integer k, let*
$$k\mathcal{P} = \{\,kp = (kp_1,\ldots,kp_d)\,|p=(p_1,\ldots,p_d)\in\mathcal{P}\}.$$

Definition 3.4 *A set $\rho\subset\mathbb{N}^d$ is called a d-dimensional basis simplex if*

- *$|\rho| = d+1$, and*
- *for all $p\in\rho$, $p_i = 0$ except possibly one $p_j\neq 0$.*

Note that if $\mathsf{Vol}_d(\rho) = 0$, then $|\mathsf{S}_{\mathbf{x}}(\theta_\rho)| = 0$, i.e., the Dixon polynomial is zero. For ρ to have a zero d-dimensional volume, it must be of dimension less than d, i.e., there must exist a coordinate, say i-th, such that for all points in ρ, their i-th coordinate is 0. In terms of monomials appearing in a polynomial system, this implies that some variable does not appear at all, and hence two rows in the expansion of the determinant for the Dixon polynomial will be the same.

If $\mathsf{Vol}_d(\rho) > 0$, then there exist d points in ρ such that each is lying on a unique coordinate axis, and the $d+1$-th point is on any axis.

Assume that $\rho = \{\alpha_1,\ldots,\alpha_d,\alpha_{d+1}\}$, and $\alpha_1,\ldots,\alpha_d$ are points lying on respective axes $x_1,\ldots,x_d$. The last point α_{d+1} is lying on the same axis as α_j, but $|\alpha_{d+1}| < |\alpha_j|$, i.e., is closer to origin. Then,

$$d!\mathsf{Vol}_d(\rho) = |\alpha_j - \alpha_{d+1}| \prod_{\substack{i=1\\i\neq j}}^{d} |\alpha_i|.$$

Proposition 3.1 *For a generic unmixed polynomial system with support ρ that is a basis simplex,*

$$|\mathsf{S}_{\mathbf{x}}(\theta_\rho)| = d!\mathsf{Vol}_d(\rho),$$

i.e. ρ is Dixon-exact with respect to any variable order.

Proof: Assume $\rho = \{\alpha_1,\ldots,\alpha_d,\alpha_{d+1}\}$ as above. The expression for the Dixon polynomial of such a generic system is given by

$$D = \begin{vmatrix} x_j^{\alpha_{d+1}} & x_1^{\alpha_1} & x_2^{\alpha_2} & \cdots & x_d^{\alpha_d} \\ x_j^{\alpha_{d+1}} & \overline{x}_1^{\alpha_1} & x_2^{\alpha_2} & \cdots & x_d^{\alpha_d} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_j^{\alpha_{d+1}} & \overline{x}_1^{\alpha_1} & \overline{x}_2^{\alpha_2} & \cdots & x_d^{\alpha_d} \\ \overline{x}_j^{\alpha_{d+1}} & \overline{x}_1^{\alpha_1} & \overline{x}_2^{\alpha_2} & \cdots & x_d^{\alpha_d} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \overline{x}_j^{\alpha_{d+1}} & \overline{x}_1^{\alpha_1} & \overline{x}_2^{\alpha_2} & \cdots & \overline{x}_d^{\alpha_d} \end{vmatrix} =$$

$$x_j^{\alpha_{d+1}}\overline{x}_j^{\alpha_{d+1}}(x_j^{\alpha_j-\alpha_{d+1}} - \overline{x}_j^{\alpha_j-\alpha_{d+1}}) \prod_{\substack{i=1\\i\neq j}}^{d} (x_i^{\alpha_i} - \overline{x}_i^{\alpha_i}).$$

The Dixon polynomial is then $CD\prod_{i=1}^{d}\frac{1}{\overline{x}_i-x_i}$, where C is the coefficient matrix. We thus have

$$\mathsf{S}_{\mathsf{x}}(\theta_\rho) = \{\, p \mid 0 \le p_i < \alpha_i \text{ for } i=1,\dots,d \text{ and } \alpha_{d+1} \le p_j < \alpha_j \,\}.$$

From above, the result follows. □

It is easy to see that $|\rho + \mathsf{S}_{\mathsf{x}}(\theta_\rho)| = (d+1)|\mathsf{S}_{\mathsf{x}}(\theta_\rho)|$. The above relation is used in the proposition 3.3 to follow.

Proposition 3.2 *Let ρ_1 and ρ_2 be two basis simplexes, such that for every $p \in \rho_1$ there exist $s, t \in \rho_2$, such that $s_i \le p_i \le t_i$ for all $i = 1,\dots,d$, then*

$$\mathsf{S}_{\mathsf{x}}(\theta_{\rho_1}) \subseteq \mathsf{S}_{\mathsf{x}}(\theta_{\rho_2}).$$

This follows from the proof of Proposition 3.1, and the fact that a Dixon polynomial is a sum of determinants corresponding to the basis simplexes as per Proposition 2.2.

If the support of a polynomial system is $\mathcal{A} = \rho_1 \cup \rho_2$ where ρ_1, ρ_2 satisfy the conditions in the above proposition, then, $\mathsf{S}_{\mathsf{x}}(\theta_{\rho_1\cup\rho_2}) = \mathsf{S}_{\mathsf{x}}(\theta_{\rho_2})$, i.e. all points of ρ_1 are Dixon-interior with respect to ρ_2.

Definition 3.5 *A support $\mathcal{A}$ is called a basis support if any $d+1$ subset of it is a basis simplex.*

Theorem 3.1 *Given a generic, unmixed polynomial system P with a basis support $\mathcal{A}$, $\mathcal{A}$ is Dixon-exact.*

The proof follows from Proposition 2.2.

Note that the one-dimensional case is a special case of the above theorem. The Dixon resultant formulation yields exact resultant in the univariate case when the degrees of the two polynomials are the same (the unmixed case).

Below, we define a geometric operation on polytopes that preserves the property of a polytope being Dixon-exact. By iterating this geometric operation on polytopes, we can obtain other polytopes.

Definition 3.6 *Given two polytopes $\mathcal{P}$ and $\mathcal{Q}$, define direct sum of $\mathcal{P}$ and $\mathcal{Q}$ to be*

$$\mathcal{P} \oplus \mathcal{Q} = \{(p_1,\dots,p_k,q_1,\dots,q_l) \mid p=(p_1,\dots,p_k) \in \mathcal{P} \subset \mathbb{N}^k, \text{ and } q=(q_1,\dots,q_l) \in \mathcal{Q} \subset \mathbb{N}^l, \}$$

where $k + l = d$.

Note $\mathcal{P} \oplus \mathcal{Q} \subset \mathbb{N}^d$, and $\mathsf{Vol}_d(\mathcal{P}\oplus\mathcal{Q}) = \mathsf{Vol}_k(\mathcal{P})\ \mathsf{Vol}_l(\mathcal{Q})$. The direct sum can be thought as the Minskowski sum where $\mathcal{P}$ and $\mathcal{Q}$ are embedded into $\mathbb{N}^d$, and added.

Proposition 3.3 *Suppose $\mathcal{A} = \mathcal{P} \oplus \mathcal{Q}$, where $\mathcal{P}, \mathcal{Q}$ are Dixon-exact using variable orders $X_{\mathcal{P}}$ and $X_{\mathcal{Q}}$, respectively. If $\mathcal{Q}$ is a basis support, then $\mathcal{A}$ is Dixon-exact with respect to the variable order $\{X_{\mathcal{P}}, X_{\mathcal{Q}}\}$.*

Proof: Using the variable order $\{X_{\mathcal{P}}, X_{\mathcal{Q}}\}$, the Dixon polynomial for the polynomial system with support $\mathcal{A}$ is:

$$\theta(f_0,\dots,f_d) = \prod_{i=1}^{d}\frac{1}{\overline{x}_i - x_i}$$

$$\left|\begin{array}{cccc}
\pi_0(f_0) & \pi_0(f_1) & \cdots & \pi_0(f_d)\\
\vdots & \vdots & \ddots & \vdots\\
\pi_{k-1}(f_0) & \pi_{k-1}(f_1) & \cdots & \pi_{k-1}(f_d)\\
\hline
\pi_k(f_0) & \pi_k(f_1) & \cdots & \pi_k(f_d)\\
\vdots & \vdots & \ddots & \vdots\\
\pi_{k+l-1}(f_0) & \pi_{k+l-1}(f_1) & \cdots & \pi_{k+l-1}(f_d)\\
\hline
\pi_{k+l}(f_0) & \pi_{k+l}(f_1) & \cdots & \pi_{k+l}(f_d)
\end{array}\right|.$$

The above matrix has been partitioned into 3 sets of rows. In the first part, only variables in $X_{\mathcal{P}}$ are replaced with variables in $X_{\mathcal{Q}}$ not getting changed; in the second part $X_{\mathcal{Q}}$ variables are changed with all variables in $X_{\mathcal{P}}$ already replaced by new variables. In the last row, all variables have been replaced by new variables.

Since the last row does not contain any original variable, it will not contribute to the support of the Dixon polynomial.

Using the Laplace formula, the above determinant can be written as the sum of products of the determinant of a minor from the upper part, the determinant of a minor with a different subset of columns from the second part, and the element in the column not considered, from the last row.

The minor from the upper part will have support which is contained in

$$\mathsf{S}_{\mathsf{x}}(\theta_{\mathcal{P}}) + \underbrace{\mathcal{Q} + \cdots + \mathcal{Q}}_{k},$$

and any minor from the lower part will have support $\mathsf{S}_{\mathsf{x}}(\theta_{\mathcal{Q}})$. Hence the support of the Dixon polynomial will be contained in the Minskowski sum

$$\mathsf{S}_{\mathsf{x}}(\theta_{\mathcal{P}}) + \underbrace{\mathcal{Q} + \cdots + \mathcal{Q}}_{k} + \mathsf{S}_{\mathsf{x}}(\theta_{\mathcal{Q}}).$$

Since $\mathcal{P}$ and $\mathcal{Q}$ are supports based on different variables,

$$|\mathsf{S}_{\mathsf{x}}(\theta_{\mathcal{A}})| = |\mathsf{S}_{\mathsf{x}}(\theta_{\mathcal{P}})||k\mathcal{Q} + \mathsf{S}_{\mathsf{x}}(\theta_{\mathcal{Q}})|.$$

Let $V_{\mathcal{Q}}$ be the extremal vertices in $\mathcal{Q}$ (obtained after removing any Dixon-interior points from $\mathcal{Q}$).

$$\text{For} \quad p, q \in \underbrace{V_{\mathcal{Q}} + \cdots + V_{\mathcal{Q}}}_{k}$$

$$p + \mathsf{S}_{\mathsf{x}}(\theta_{\mathcal{Q}}) \cap q + \mathsf{S}_{\mathsf{x}}(\theta_{\mathcal{Q}}) = \emptyset,$$

because the maximum of any point coordinate in $\mathsf{S}_{\mathsf{x}}(\theta_{\mathcal{Q}})$ is smaller than the corresponding maximum in $V_{\mathcal{Q}}$. Then,

$$|\mathsf{S}_{\mathsf{x}}(\theta_{\mathcal{A}})| = |\mathsf{S}_{\mathsf{x}}(\theta_{\mathcal{P}})||\underbrace{V_{\mathcal{Q}} + \cdots + V_{\mathcal{Q}}}_{k}||\mathsf{S}_{\mathsf{x}}(\theta_{\mathcal{Q}})| =$$

$$\binom{k+l}{k} k!\mathsf{Vol}_d(\mathcal{P}) l!\mathsf{Vol}_d(\mathcal{Q}) = d!\mathsf{Vol}_d(\mathcal{A}).$$

This implies that the support $\mathcal{A}$ is Dixon-exact. □

In the above proof, the following property is used:

Proposition 3.4 *Let $V_{\mathcal{Q}}$ be a support consisting only of extremal points (i.e. $\mathcal{Q}$ without interior points), then*

$$|\underbrace{V_{\mathcal{Q}} + \cdots + V_{\mathcal{Q}}}_{k}| = \binom{d+k}{k}.$$

Proof: By induction on the dimension d. The basis case is $d=1$. Then V_Q is just 2 points on a line, and

$$|\underbrace{V_Q+\cdots+V_Q}_{k}|=k+1.$$

Assume, the proposition is true for d. $V_Q=V_{Q'}\cup p$ where $V_{Q'}$ has d points, where all have one coordinate value 0, and p has non-zero value in that coordinate. Let

$$S_j=jp+\underbrace{V_{Q'}+\cdots+V_{Q'}}_{k-j}\quad\text{for}\quad j=0,\ldots,k.$$

Note that $V_Q+\cdots+V_Q=\cup_{j=0}^{k}S_j$. Also note that $S_j\cap S_i=\oslash$ for $j\neq i$. Using the induction hypothesis,

$$\begin{aligned}|\underbrace{V_Q+\cdots+V_Q}_{k}| &= \sum_{j=0}^{k}\binom{d+k-j}{k-j}=\sum_{i=0}^{k}\binom{d+i}{i}\\ &= \binom{d+1+k}{k}.\square\end{aligned}$$

Theorem 3.2 *Given a generic unmixed polynomial system whose support is a direct sum of k Dixon-exact supports $\mathcal{A}_1\oplus\cdots\oplus\mathcal{A}_k$, where $\mathcal{A}_i$, for $1\leq i\leq k$, is a basis support, the Dixon formulation computes the exact resultant.*

The proof follows by induction from Proposition 3.3.

3.1 n-Degree Systems

Definition 3.7 *A support $\mathcal{A}$ is n-degree if there exists non-negative integers $k_1,\ldots,k_d$ such that $\mathcal{A}=\{\,p\mid 0\leq p_i\leq k_i\}$.*

Proposition 3.5 *A n-degree support $\mathcal{A}$ can be expressed as $\mathcal{A}=\mathcal{X}_1\oplus\cdots\oplus\mathcal{X}_d$, where $\mathcal{X}_i$ is 1-dimensional basis support and each $p\in\mathcal{X}_i$ satisfies $0\leq p\leq k_i$.*

For generic unmixed n-degree polynomial systems, the Dixon formulation has been shown to compute exact resultants (see [13]). Here we show that this result is a special case of Theorem 3.2.

Corollary 3.5.1 *The Dixon resultant formulation yields the exact resultant for a generic unmixed n-degree polynomial system.*

Proof: A generic unmixed n-degree polynomial system has an n-degree support, which by the above proposition, can be expressed as a direct sum of Dixon-exact supports. The statement follows from Theorem 3.2. $\square$

4 Multihomogeneous Systems

Definition 4.1 *A polynomial f is called multihomogeneous of type $(l_1,l_2,\ldots,l_r;k_1,k_2,\ldots,k_r)$ if*

$$f=\sum c_\alpha \mathbf{x}_1^{k_1}\mathbf{x}_2^{k_2}\cdots\mathbf{x}_r^{k_r},\ \textit{where}\ \mathbf{x}_i^{k_i}=x_{i,1}^{p_1}\cdots x_{i,l_{i+1}}^{p_{l_{i+1}}},$$
$$\textit{and}\qquad p_1+p_2+\cdots p_{l_i}+p_{l_{i+1}}=k_i.$$

That is, each block has a homogenizing variable. A polynomial is called multihomogeneous of some type, if it can be homogenized into such a multihomogeneous polynomial. Note that a polynomial can be made multihomogeneous in a number of ways, depending on the partition of variables. We will call the support of a multihomogeneous polynomial to be multihomogeneous as well.

Proposition 4.1 *If a support $\mathcal{A}$ is multihomogeneous of type $(l_1,l_2,\ldots,l_r;k_1,k_2,\ldots,k_r)$, then it can be written as*

$$\mathcal{A}=\mathcal{X}_1\oplus\cdots\oplus\mathcal{X}_r,$$

where $\mathcal{X}_i$ is the support of a dense polynomial of total degree of k_i, in terms of the variables of i^{th} block.

4.1 Multigraded systems

Definition 4.2 *A multihomogeneous system of type $(l_1,l_2,\ldots,l_r;k_1,k_2,\ldots,k_r)$ is called **multigraded** if $\forall i=1,\ldots,r$ either $l_i=1$ or $k_i=1$.*

Multigraded systems were introduced in [15] as a special case of multihomogeneous systems.

Multihomogeneous supports are direct sum of smaller supports as shown above, yet they are not necessarily Dixon-exact. But multigraded systems are.

Proposition 4.2 *A multigraded support $\mathcal{A}$ can be expressed as $\mathcal{X}_1\oplus\cdots\oplus\mathcal{X}_r$, where $\mathcal{X}_i$ is Dixon-exact for all $i=1,\ldots,r$.*

Proof: When $l_i=1$ for some $1\leq i\leq r$, $\mathcal{X}_i$ is one dimensional support and hence is Dixon-exact.

Whenever $k_i=1$ for some $1\leq i\leq r$, $\mathcal{X}_i$ consists of a zero l_i-tuple, and l_i-tuples in which exactly one coordinate is non-zero, and has value 1. $\mathcal{X}_i$ is then a basis simplex, which is Dixon-exact. $\square$

From the above Proposition and Theorem 3.2, it follows:

Corollary 4.2.1 *The Dixon resultant formulation yields exact resultant for a generic unmixed multigraded polynomial system.*

We would like to point out that not all Dixon-exact supports are multigraded, for example: suppose we are given unmixed generic polynomial system where each polynomial has $\{1,y^2,z,x^3,x^3y^2,x^3z,x^2y\}$ as monomials. Clearly this is not multigraded polynomial system, yet it is Dixon exact since its support is direct sum of $1,y^2,z$ and $1,x^3$ where monomial x^2y is Dixon-interior.

Theorem 3.2 in the previous section is, thus, a generalization of the theorem in [15] where it is proved that for multigraded systems, Sylvester-type resultant formulas can be given.

As pointed out in [15], there is $r!$ resultant matrices for multigraded whose determinant is the resultant. It can be shown that the Dixon resultant formulation can construct any of $r!$ exact matrices as well, depending on the order of given blocks used in the construction. The order within blocks will change only the monomial multipliers.

Figure 1 depicts the support of a multigraded system of type $(1,2;2,1)$, where variable blocks are $\{x\}$ and $\{y,z\}$. Under variable orders $\{x,y,z\},\{x,z,y\},\{y,z,x\}$ and $\{z,y,x\}$ Dixon resultant formulation yields 6×6 Dixon matrix or 24×24 sparse resultant matrix from which exact resultant is extracted in accordance with theorem 3.2.

Yet variable order $\{y,x,z\}$ or $\{z,x,y\}$ give rise to Dixon matrix of size 8×8 or sparse matrix of size 32×30, hence resulting in extra factor of degree 2 in the coefficients of any one polynomial in the polynomial system.

Consider a multihomogeneous system: for $i=0,1,2$:

$$c_{i,00}+c_{i,01}y+c_{i,02}y^2+c_{i,10}x+c_{i,11}xy+c_{i,20}x^2$$

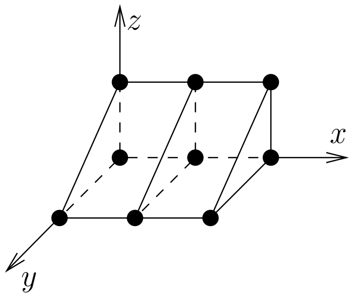

Figure 1: Multigraded system of type $(1, 2; 2, 1)$

It can be made multihomogeneous of type $(1, 1; 2, 2)$, in which case it is multigraded, but then not generic, since its support does not contain all vertices that a system of type $(1, 1; 2, 2)$ can have, in particular monomial x^2y^2 is missing. But if this system is viewed as multihomogeneous of type $(2; 2)$, then all monomial are present.

This example illustrates that it is not always possible to construct exact sparse resultant matrices. The above system is unmixed, and the mixed volume of any two polynomials is 4. Hence the resultant matrix has at least 4 rows coming from each polynomial, to imply that it must have 12 rows to be exact resultant matrix, and also 12 columns.

The smallest matrix one can construct is 12×13. One can check it by brute force, trying to arrange 4 copies of each the above polynomial supports on a 12 point polytope. But even then such matrix is not a resultant matrix. The smallest we came up with is 14×14. Dixon matrix is 5×5 and has extraneous factor of degree 1 in coefficients of any one of the equations. Note that Macaulay extracts the exact resultant in this by finding sub-matrix whose determinant is the extra factor, and hence the resultant can be retrieved. But the question in general whether there is always such a sub-matrix is still a open question.

5 Bivariate Case

In this section, we discuss the case of $d = 2$, and give a precise characterization of when the Dixon resultant formulation leads to exact resultants and when extraneous factors are generated.

Let $\{\alpha, \beta, \gamma\}$ be a simplex in two dimensions. We will associate with it, the following two determinants:

$$|\alpha,\beta,\gamma| = \prod_{i=1}^{d} \frac{1}{\overline{x}_i - x_i} \begin{vmatrix} x^{\alpha_x}y^{\alpha_y} & x^{\beta_x}y^{\beta_y} & x^{\gamma_x}y^{\gamma_y} \\ \overline{x}^{\alpha_x}y^{\alpha_y} & \overline{x}^{\beta_x}y^{\beta_y} & \overline{x}^{\gamma_x}y^{\gamma_y} \\ \overline{x}^{\alpha_x}\overline{y}^{\alpha_y} & \overline{x}^{\beta_x}\overline{y}^{\beta_y} & \overline{x}^{\gamma_x}\overline{y}^{\gamma_y} \end{vmatrix},$$

$$2\,\mathsf{Vol}(\alpha,\beta,\gamma) = \begin{vmatrix} 1 & \alpha_x & \alpha_y \\ 1 & \beta_x & \beta_y \\ 1 & \gamma_x & \gamma_y \end{vmatrix}.$$

In the expression for $|\alpha, \beta, \gamma|$, the total degree in $\overline{x}, x$ is $\alpha_x + \beta_x + \gamma_x - 1$, and it is the same for $\overline{y}, y$ as well. To simplify the analysis of the monomial set of $|\alpha, \beta, \gamma|$ and without any loss of generality, substitute $\overline{x} = 1, \overline{y} = 1$.

Consider a support $\mathcal{A} = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and $\min_{i=1}^{n}(\alpha_i)_x < m < \max_{i=1}^{n}(\alpha_i)_x$. Consider the following two maps: $\psi^{\pm}_{x=m} : \mathbb{N}^d \to \mathbb{N}^d$ where for $a = (a_x, a_y) \in \mathbb{N}^2$,

$$\psi^{-}_{x=m}((a_x, a_y)) = \begin{cases} (m, a_y) & a_x > m, \\ (a_x, a_y) & a_x \leq m, \end{cases}$$

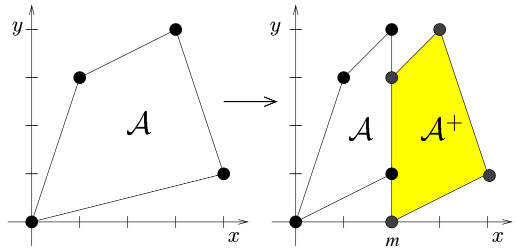

Figure 2: $\psi_{x=m}$ Operation on support $\mathcal{A}$

and also

$$\psi^{+}_{x=m}((a_x, a_y)) = \begin{cases} (m, a_y) & a_x < m, \\ (a_x, a_y) & a_x \geq m. \end{cases}$$

See Figure 2 for an example. These maps can be looked as maps on individual terms, where

$$\psi^{\pm}_{x=m}(\mathbf{x}^{\alpha}) = \mathbf{x}^{\psi^{\pm}_{x=m}(\alpha)},$$

or even on an entire polynomial, in which case ψ is applied to each term in the polynomial.

Let $\mathcal{A}^- = \psi^{-}_{x=m}(\mathcal{A})$ and $\mathcal{A}^+ = \psi^{+}_{x=m}(\mathcal{A})$. Note that $\mathcal{A}^- = S_{\mathbf{x}}(\psi^{-}_{x=m}(P_{\mathcal{A}}))$, where $P_{\mathcal{A}}$ is a generic polynomial whose support is $\mathcal{A}$. Abusing the notation, define for any $\alpha \in \mathcal{A}$, $\alpha^- = \psi^{-}_{x=m}(\alpha)$.

Note that $\mathsf{S}_{\mathbf{x}}(\theta_{\mathcal{A}^-}) = \mathsf{S}_{\mathbf{x}}(\psi^{-}_{x=m}(\theta_{\mathcal{A}}))$ even though $\theta_{\mathcal{A}^-} \neq \psi^{-}_{x=m}(\theta_{\mathcal{A}})$. To see this, note that the above map $\psi^{\pm}_{x=m}$ will result in "splitting" various determinants in the Dixon polynomial (expressed using Proposition 2.2). It thus suffices to consider one of them. Below, we assume that $\alpha_x \leq \beta_x \leq m \leq \gamma_x$ and $\Delta = (1 - x)(1 - y)$.

$$\frac{1}{\Delta}\begin{vmatrix} x^{\alpha_x}y^{\alpha_y} & x^{\beta_x}y^{\beta_y} & x^{\gamma_x}y^{\gamma_y} \\ y^{\alpha_y} & y^{\beta_y} & y^{\gamma_y} \\ 1 & 1 & 1 \end{vmatrix} =$$

$$\frac{1}{\Delta}\begin{vmatrix} x^{\alpha_x}y^{\alpha_y} & x^{\beta_x}y^{\beta_y} & x^{m}y^{\gamma_y} \\ y^{\alpha_y} & y^{\beta_y} & y^{\gamma_y} \\ 1 & 1 & 1 \end{vmatrix} + \frac{1}{\Delta}\begin{vmatrix} x^{m}y^{\alpha_y} & x^{m}y^{\beta_y} & x^{\gamma_x}y^{\gamma_y} \\ y^{\alpha_y} & y^{\beta_y} & y^{\gamma_y} \\ 1 & 1 & 1 \end{vmatrix},$$

which splits into two disjoint monomial sets, since in the first determinant, $\deg(x) \leq m$, and all monomials of the second determinant have x^m as a factor. In the expression for the Dixon polynomial, these determinants are divided by $(1 - x)(1 - y)$, after which in the first determinant, the degree of $x < m$, whereas in the second determinant, the degree of $x \geq m$.

The following two identities hold:

- $\theta_{\mathcal{A}} = \psi^{-}_{x=m}(\theta_{\mathcal{A}}) + \psi^{+}_{x=m}(\theta_{\mathcal{A}})$, hence
$$\mathsf{S}_{\mathbf{x}}(\theta_{\mathcal{A}}) = \mathsf{S}_{\mathbf{x}}(\theta_{\mathcal{A}^-}) \cup \mathsf{S}_{\mathbf{x}}(\theta_{\mathcal{A}^+}).$$
- $\mathsf{S}_{\mathbf{x}}(\theta_{\mathcal{A}^-}) \cap \mathsf{S}_{\mathbf{x}}(\theta_{\mathcal{A}^+}) = \emptyset$.

We can also split the support on y. Define a map

$$\psi^{-}_{y=m}((a_x, a_y)) = \begin{cases} (a_x, m) & a_y > m, \\ (a_x, a_y) & a_y \leq m, \end{cases}$$

for $a = (a_x, a_y) \in \mathbb{N}^2$, and also

$$\psi^{+}_{y=m}((a_x, a_y)) = \begin{cases} (a_x, m) & a_y < m, \\ (a_x, a_y) & a_y \geq m. \end{cases}$$

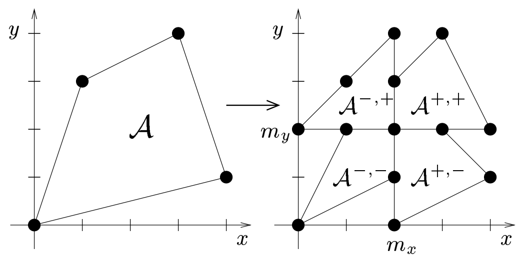

Figure 3: $\psi_{\substack{x=2\\y=2}}$ Operation on support $\mathcal{A}$

Below assume that $\alpha_y \le \beta_y \le m \le \gamma_y$.

$$\frac{1}{\Delta}|\alpha\beta\gamma| = \frac{1}{\Delta}y^{\gamma_y - m}|\alpha'\beta'\gamma| + \frac{1}{\Delta}y^{\alpha_y+\beta_y-2m}|\alpha\beta\gamma'|$$

where $\alpha' = (\alpha_x, m)$, $\beta' = (\beta_x, m)$ and $\gamma' = (\gamma_x, m)$. By a similar argument as for x, the two determinant in the sum have all disjoint monomials in y.

Maps $\psi^{\pm}_{y=m}$ can be viewed as partitioning maps for the support of the polynomial system, as well as the support of its Dixon polynomial in a same way as $\psi^{\pm}_{x=m}$. To make $\psi^{\pm}_{y=m}$ partitioning, we had to premultiply individual determinants in above expression.

Two maps $\psi^{\pm}_{x=m_x}$ and $\psi^{\pm}_{y=m_y}$ can be composed together, and the composition is commutative. By $\psi^{+,-}_{2,2}$ we will denote $\psi^{+}_{x=2} \circ \psi^{-}_{y=2}$. See Figure 3 where the polytope is first split on $x = 2$, and then each resulting polytope is split again on $y = 2$.

Given a support $\mathcal{A}$ of a polynomial system P, we can apply ψ map on every point belonging to the convex hull of $\mathcal{A}$. After partitioning $\mathcal{A}$ into the smallest polytopes (i.e. when they cannot be partitioned any further), each resulting polytope can be shown to be Dixon-exact, and hence the sum of the volumes of the smaller polytopes is the number of monomials in the Dixon polynomial.

Proposition 5.1 *A point $p \in \mathcal{A}$ is Dixon-interior if there exists $a, b, c, d \in \mathcal{A}$ different from p such that*

$$\{a_y, d_y\} \le p_y \le \{b_y, c_y\} \quad \textit{and} \quad \{a_x, b_x\} \le p_x \le \{d_x, c_x\}.$$

Proof: Since for all maps $\psi^{\pm}_{\substack{x=p_x\\y=p_y}}(q) = (p_x, p_y)$, for $q \in \{a, b, c, d\}$, the presence of p in $\mathcal{A}$ after partitioning is irrelevant, i.e. it does not influence the size of the support of the Dixon polynomial. □

As should be evident from maps ψ, Dixon-interior points do not contribute to the support of the Dixon polynomial, and hence the resultant. When ψ map is applied on the support as much as possible, Dixon-interior points in the support will be overlapped by other points in the support.

An analysis of how the map ψ changes polytope volume, indicates cases when the support of the Dixon polynomial has more monomials than the mixed volume of the polytopes, and hence cases, when the Dixon resultant formulation yields extraneous factors.

Definition 5.1 *A 2-dimensional simplex $\rho = \{\alpha, \beta, \gamma\}$ is orderable if β is strictly in between α, γ in every coordinate.*

Proposition 5.2 *A simplex $\rho = \{\alpha, \beta, \gamma\} \subseteq \mathcal{A}$ is Dixon-exact if and only if it is not orderable.*

Proof: ψ map preserves the number of monomials in a Dixon polynomial. A simplex whose volume is not invariant under ψ, has more monomials than its volume, and hence, is not Dixon-exact. Conversely, if the volume of a simplex is invariant under ψ, that simplex is Dixon-exact.

Without any loss of generality, assume that $\alpha_x \le \beta_x \le m \le \gamma_x$. Let $\rho^- = \psi^{-}_{x=m}(\rho)$ and $\rho^+ = \psi^{+}_{x=m}(\rho)$. Then

$$\begin{vmatrix} 1 & \alpha_x & \alpha_y \\ 1 & \beta_x & \beta_y \\ 1 & \gamma_x & \gamma_y \end{vmatrix} = \begin{vmatrix} 1 & \alpha_x & \alpha_y \\ 1 & \beta_x & \beta_y \\ 1 & m & \gamma_y \end{vmatrix} + \begin{vmatrix} 1 & m & \alpha_y \\ 1 & m & \beta_y \\ 1 & \gamma_x & \gamma_y \end{vmatrix}.$$

which is exactly $2\ \mathsf{Vol}(\rho) = 2\ \mathsf{Vol}(\rho^-) + 2\ \mathsf{Vol}(\rho^+)$.

It suffices to show that each Vol in the above expression has the same sign for any value of m, and hence volume is invariant under ψ. At the same time we will derive the condition when volumes have different signs, and hence ψ does not preserve volume.

Suppose that one of the determinants on the right side has a sign different from the sign of the determinant on the left side. If the first determinant has a different sign, we can look at the transformation from the left determinant to the first determinant as a continuous function. Hence, $\exists w$ in $m < w < \gamma_x$ such that

$$\begin{vmatrix} 1 & \alpha_x & \alpha_y \\ 1 & \beta_x & \beta_y \\ 1 & w & \gamma_y \end{vmatrix} = 0, \quad \text{i.e.,} \quad \begin{array}{l} (\beta_x - \alpha_x)(\gamma_y - \beta_y) - \\ \qquad (w - \beta_x)(\beta_y - \alpha_y) = 0 \end{array}$$

Note $(\beta_x - \alpha_x) \ge 0$ and $(w - \beta_x) \ge 0$, yet for $w \to m$, this changes the sign, implying that $\gamma_y > \beta_y > \alpha_y$ or $\alpha_y < \beta_y < \gamma_y$ together with $\alpha_x < \beta_x < \gamma_x$.

If the second determinant on the right side changes the sign, then there is $w_1 \le w_2$ such that $\alpha_x \le w_1 \le m$ and $\beta_x \le w_2 \le m$, and

$$\begin{vmatrix} 1 & w_1 & \alpha_y \\ 1 & w_2 & \beta_y \\ 1 & \gamma_x & \gamma_y \end{vmatrix} = 0, \quad \text{i.e.,} \quad \begin{array}{l} (w_2 - w_1)(\gamma_y - \beta_y) - \\ \qquad (\gamma_x - w_2)(\beta_y - \alpha_y) = 0. \end{array}$$

Both $w_2 - w_1 \ge 0$ and $\gamma_x - w_2 \ge 0$; as $w_1 \to m$, there is a sign change, implying $\alpha_y < \beta_y < \gamma_y$ or $\gamma_y > \beta_y > \alpha_y$. □

Proposition 5.3 *Let support $\mathcal{A} = \{\alpha, \beta, \gamma, \delta\}$ such that $\alpha_x \le \beta_x \le \gamma_x \le \delta_x$. If every simplex of $\mathcal{A}$ is Dixon-exact, then $\mathcal{A}$ is Dixon-exact.*

Proof: It can be shown that

$$|\alpha, \beta, \gamma| + |\gamma, \beta, \delta| = |\beta, \alpha, \delta| + |\alpha, \gamma, \delta| \quad \text{and}$$

$$\mathsf{Vol}(\mathcal{A}) = \mathsf{Vol}(\alpha, \beta, \gamma) + \mathsf{Vol}(\gamma, \beta, \delta).$$

Since each simplex is Dixon-exact, $\mathcal{A}$ is Dixon-exact. □.

Proposition 5.4 *Let $\mathcal{A}' = \mathcal{A} - I_{\mathcal{A}}$, where $I_{\mathcal{A}}$ is the set of Dixon interior points, then $\mathcal{A}'$ and hence $\mathcal{A}$ is Dixon exact if every simplex of $\mathcal{A}'$ is Dixon-exact.*

Proof: From Proposition 2.2, $\mathsf{S_x}(\theta_{\mathcal{A}'}) = \bigcup_{\rho \subseteq \mathcal{A}'} \mathsf{S_x}(|\rho_1\rho_2\rho_3|)$. Call two simplexes ρ_1, ρ_2 disjoint if $\mathsf{Vol}(\rho_1) + \mathsf{Vol}(\rho_2) = Vol(\rho_1 \cup \rho_2)$. Let $T_{\mathcal{A}'}$ be the set of disjoint simplexes of $\mathcal{A}'$. Below we show that

$$\mathsf{S_x}(\theta_{\mathcal{A}'}) = \bigcup_{\rho \in T_{\mathcal{A}'}} \mathsf{S_x}(|\rho_1\rho_2\rho_3|),$$

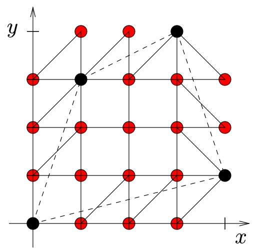

Figure 4: Complete partition of support $\mathcal{A}$

from which the statement follows.

The above is true when $|\mathcal{A}'| = 3$ or 4. Using them as the bases, the proof is by induction on $|\mathcal{A}'|$. Assume that it is true for a support of size n, and let $|\mathcal{A}'| = n+1$ and $\mathcal{A}' = \mathcal{Q} \cup \{q\}$.

$$\mathsf{S_x}(\theta_{\mathcal{A}'}) = \bigcup_{\rho \in T_{\mathcal{Q}}} \mathsf{S_x}(|\rho_1\rho_2\rho_3|) \cup \bigcup_{a,b \in \mathcal{Q}} \mathsf{S_x}(|a,b,q|).$$

If $|a,b,q|$ and $|a,b,c| \in T_{\mathcal{Q}}$ are not disjoint, then $|a,b,q|$ can be deleted since $|a,b,q| \subseteq |a,b,c| \cup |s,q,c|$ where $s \in \{a,b\}$ by the same argument as in Proposition 5.3. Hence only disjoint simplexes need to be considered.

Theorem 5.1 *Given a generic unmixed polynomial system with support $\mathcal{A}$, $\mathcal{A}$ not including an orderable simplex is necessary and sufficient condition for the Dixon formulation to compute its exact resultant.*

5.1 Example

Revisiting the example in Figure 2, after partitioning the support completely we get the set of supports shown in Figure 4. Note that each support in the partition is Dixon-exact. Their total volume is 19, and hence there will be 19 monomial in the Dixon polynomial. But $2\ \mathsf{Vol}_d(\mathcal{A}) = 18$, hence the Dixon matrix will yield an extra factor of degree 1 in the coefficients of each polynomial in the system.

After applying $\psi_{\substack{x=2\\y=2}}$ on the support, the total volume is still 18. It can be seen that the extra monomial appears in $\mathcal{A}^{-,+}$, which results from points $(0,0), (1,3), (3,4)$, which is exactly the case. Proposition 5.2 predicts this condition.

6 Conclusion

In this paper, we have identified supports of unmixed generic polynomial systems for which the Dixon resultant formulation computes exact resultants. The main result is a generalization of the results reported in the literature about n-degree generic unmixed polynomial systems as well as about generic unmixed multigraded systems. For the bivariate case, exact analysis is given whereby it can be determined for which unmixed generic polynomial systems, the Dixon resultant formulation computes exact resultants, and because of which monomials in an unmixed generic polynomial system, the Dixon resultant formulation produces extraneous factors in resultant computations.

We are exploring ways to generalize the analysis for the bivariate case to the general d-dimensional case.

References

[1] Becker, T., Weispfenning, V., and Kredel, H. *Groebner Bases - A Computational Approach to Commutative Algebra.* Springer Verlag, 1993.

[2] Buchberger, B. Groebner bases: An algorithmic method in polynomial ideal theory. *In: (ed. N.K. Bose) Multidimensional System Theory* (1985), 184–232.

[3] Canny, J., and Emiris, I. An efficient algorithm for the sparse mixed resultant. In *Proc. Intern. Symp. Applied Algebra, Algebraic Algor. and Error-Corr. Codes, Lect. Notes in Comp. Science 263* (Puerto Rico, May 1993), G. Cohen, T. Mora, and O. Moreno, Eds., Springer Verlag, pp. 89–104.

[4] Chou, S.-C., and Gao, X.-S. Methods for mechanical geometry formula deriving. *Proc. ISSAC 1990* (1990).

[5] Chthcerba, A. D., and Kapur, D. Extracting sparse resultant matrices from Dixon resultant formultation. *Accepted for presentation at RCWA'00* (2000).

[6] Dixon, A. The eliminant of three quantics in two independent variables. *Proc. London Mathematical Society 6* (1908), 468–478.

[7] Elkali, M., and Mourrain, B. Some applications of bezoutians in effective algebraic geometry. Tech. Rep. 3572, INRIA, Sophia-Antipolis, France, Dec. 1998.

[8] Gelfand, I., Kapranov, M., and Zelevinsky, A. *Discriminants, Resultants and Multidimensional Determinants*, first ed. Birkhauser, Boston, 1994.

[9] Kapur, D., and Saxena, T. Comparison of various multivariate resultants. In *ACM ISSAC 95* (Montreal, Canada, July 1995).

[10] Kapur, D., and Saxena, T. Sparsity considerations in the Dixon resultant formulation. In *Proc. ACM Symposium on Theory of Computing* (Philadelphia, May 1996).

[11] Kapur, D., and Saxena, T. Extraneous factors in the Dixon resultant formulation. In *ISSAC* (Maui, Hawaii, USA, 1997), pp. 141–147.

[12] Kapur, D., Saxena, T., and Yang, L. Algebraic and geometric reasoning using Dixon resultants. In *ACM ISSAC 94* (Oxford, England, July 1994), pp. 99–107.

[13] Saxena, T. *Efficient variable elimination using resultants.* PhD thesis, Department of Computer Science, State Univeristy of New York, Albany, NY, 1997.

[14] Sturmfels, B. Sparse elimination theory. In *Proc. Computat. Algebraic Geom. and Commut. Algebra* (Cortona,Italy, June 1991), D. Eisenbud and e. L. Robbiano, Eds., Cambridge Univ. Press.

[15] Sturmfels, B., and Zelevinski, A. Multigraded resultants of sylvester type. *Journal of Algebra 163* (1994), 115–127.