Axiom Systems for Boolean Algebra Using the Sheffer Stroke

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Abstract

In this article, we present several new 2-axiom systems for Boolean algebra using the Sheffer stroke. In each case, we indicate how the given system can be used to derive the system presented by Sheffer in 1913 (Sheffer[5]). Our search for proofs relied heavily on the use of an automated reasoning program.

1 Introduction

In 1913, Sheffer [5] presented a 3-axiom basis for Boolean algebra using the "Sheffer stroke".

$$f(f(x,x),f(x,x)) = x \qquad \text{(Sheffer 1)}$$
$$f(x,f(y,f(y,y))) = f(x,x) \qquad \text{(Sheffer 2)}$$
$$f(f(x,f(y,z)),f(x,f(y,z))) = f(f(f(y,y),x),f(f(z,z),x)) \qquad \text{(Sheffer 3)}$$

More recently, a number of simplifications ("abridgements") of Sheffer's system have been published. These include, for example, five systems presented in Meredith[3][1],

$$f(f(x,x),f(y,x)) = x \qquad (\text{Meredith}_1\ 1)$$
$$f(f(x,f(y,z)),f(x,f(y,z))) = f(f(f(z,x),x),f(f(y,x),x)) \qquad (\text{Meredith}_1\ 2)$$

$$f(f(x,x),f(f(y,y),y)) = x \qquad (\text{Meredith}_2\ 1)$$
$$f(x,f(y,z)) = f(f(f(f(z,z),x),f(f(y,y),x)),f(f(f(z,z),x),f(f(y,y),x))) \qquad (\text{Meredith}_2\ 2)$$

$$f(f(x,x),f(x,y)) = x \qquad (\text{Meredith}_{2'}\ 1)$$
$$f(f(x,f(x,x)),f(y,f(z,u))) = f(f(f(u,u),y),f(f(z,z),y)) \qquad (\text{Meredith}_{2'}\ 2)$$

$$f(f(x,x),f(y,x)) = x \qquad (\text{Meredith}_3\ 1)$$
$$f(x,f(y,f(x,z))) = f(f(f(z,y),y),x) \qquad (\text{Meredith}_3\ 2)$$

$$f(f(x,x),f(y,x)) = x \qquad (\text{Meredith}_4\ 1)$$
$$f(x,f(y,f(x,z))) = f(x,f(y,f(y,z))) \qquad (\text{Meredith}_4\ 2)$$
$$f(x,y) = f(y,x) \qquad (\text{Meredith}_4\ 3)$$

[1]The second system, Meredith$_2$, is attributed to G. Spencer Brown.

and the following system from David Hillman.

$$f(f(x,x),f(x,y)) = x \tag{Hillman 1}$$
$$f(x,f(x,y)) = f(x,f(y,y)) \tag{Hillman 2}$$
$$f(x,f(x,f(y,z))) = f(y,f(y,f(x,z))) \tag{Hillman 3}$$

Stephen Wolfram and David Hillman have proposed a study of twenty seven candidate axiom systems consisting of twenty five single axioms and two pairs of axioms.

1. $f(f(x,f(x,f(x,y))),f(y,f(x,z))) = y$
2. $f(f(x,f(x,f(x,y))),f(y,f(z,x))) = y$
3. $f(f(x,f(x,f(y,x))),f(y,f(x,z))) = y$
4. $f(f(x,f(x,f(y,x))),f(y,f(z,x))) = y$
5. $f(f(x,f(x,f(y,y))),f(y,f(x,z))) = y$
6. $f(f(x,f(x,f(y,y))),f(y,f(z,x))) = y$
7. $f(f(x,f(x,f(y,z))),f(y,f(z,x))) = y$
8. $f(f(x,f(x,f(y,z))),f(z,f(x,y))) = z$
9. $f(f(x,f(f(y,x),x)),f(y,f(x,z))) = y$
10. $f(f(x,f(f(y,x),x)),f(y,f(z,x))) = y$
11. $f(f(x,f(f(y,z),x)),f(y,f(z,x))) = y$
12. $f(f(f(x,y),z),f(x,f(x,f(z,x)))) = z$
13. $f(f(f(x,y),z),f(x,f(x,f(z,y)))) = z$
14. $f(f(f(x,y),z),f(x,f(f(x,z),x))) = z$
15. $f(f(f(x,y),z),f(x,f(f(y,z),x))) = z$
16. $f(f(f(x,y),z),f(x,f(f(z,x),x))) = z$
17. $f(f(f(x,y),z),f(x,f(f(z,y),x))) = z$
18. $f(f(f(x,y),z),f(x,f(f(z,z),x))) = z$
19. $f(f(f(x,y),z),f(y,f(y,f(z,x)))) = z$
20. $f(f(f(x,y),z),f(y,f(y,f(z,y)))) = z$
21. $f(f(f(x,y),z),f(y,f(f(x,z),y))) = z$
22. $f(f(f(x,y),z),f(y,f(f(y,z),y))) = z$
23. $f(f(f(x,y),z),f(y,f(f(z,x),y))) = z$
24. $f(f(f(x,y),z),f(y,f(f(z,y),y))) = z$
25. $f(f(f(x,y),z),f(y,f(f(z,z),y))) = z$

26a. $f(f(x,y),f(x,f(y,z))) = x$
26b. $f(x,y) = f(y,x)$ (Commutativity)

27a. $f(f(x,z),f(x,f(y,z))) = x$
27b. $f(x,y) = f(y,x)$ (Commutativity)

In this article, we present some results from our study of the Wolfram-Hillman candidates, including several new 2-axiom bases for Boolean algebra using the Sheffer stroke. The independence of these new axiom systems remains unknown to us. It also remains unknown whether any one of the twenty five single-axiom candidates or either of the 2-axiom candidates forms a basis.

2 Notation and Conventions

The new axiom systems were identified with the assistance of the automated reasoning program Otter (McCune[2]). In the language of Otter, each equation is represented as a *clause* consisting of a single *literal* of the form `t1 = t2`, where `t1` and `t2` are *terms* (Chang[1]). Terms and subterms of clauses can be referenced using a vector notation,

```
C.L.a1.a2.a3.a4 ...
```

where `C` is a clause number; `L` is a *literal* number;[2] `a1` refers to an argument of the literal—in this case, either 1 for the left side of an equation or 2 for the right side; `a2` refers to an argument of `C.L.a1`; `a3` refers to an argument of `C.L.a1.a2`; and so on.

The inference rule employed in this study is *paramodulation* (Robinson[4]), which combines variable substitution (instantiation) and equality substitution for clauses. In addition, some equations are "flipped" as an application of symmetry of equality.

Example. Consider the following two clauses.

```
1. f(f(x,c1),c2) = f(f(c3,f(x,c4)),f(c5,x))
2. f(a,y) = f(y,a)
```

where `x` and `y` are variable symbols, and the `ci` are constant symbols. The clause

```
3. f(f(a,c1),c2) = f(f(c3,f(c4,a)),f(c5,a))
```

results from paramodulating *from* term `2.1.1` (term `f(a,y)` of clause `2`) *into* term `1.1.2.1.2` (term `f(x,c4)` of clause `1`). Specifically, the from and into terms are *unified* under the substitution { `a/x, c4/y` }, and then the designated equality substitution takes place between the two respective instances of the original clauses:

```
1'. f(f(a,c1),c2) = f(f(c3,f(a,c4)),f(c5,a))
2'. f(a,c4) = f(c4,a)
```

This derivation is represented either as `[2.1.1` $\rightarrow$ `1.1.2.1.2]` or as `[1.1.2.1.2` $\leftarrow$ `2.1.1]`, where the arrow indicates the direction of the equality substitution. (We could restrict ourselves to using one form or the other, but we make use of both forms to be consistent with Otter's presentation of proofs.)

Finding the proofs described in this article required sequences of carefully designed Otter experiments. Some of the methods used are described in Veroff[6].

[2]In general, a clause may consist of a finite disjunction of literals. In this study, however, each clause consists of a single literal, representing a single equation, so the literal number always is 1.

3 Results

In this section, we present several new 2-axiom systems for Boolean algebra using the Sheffer stroke. In each case, the proof consists of a derivation of the axioms of some known system.

Theorem 1. Axiom 7 and Commutativity together form a basis.

Proof. It suffices to derive the axioms of the Sheffer system. These axioms are flagged with a * symbol in the following proof.

1. $f(f(x,f(x,f(y,z))),f(y,f(z,x))) = y$ (Axiom 7)
2. $f(x,y) = f(y,x)$ (Commutativity)
3. $f(f(f(x,f(y,z)),f(f(x,f(y,z)),f(u,f(z,f(z,f(x,y)))))),f(u,x)) = u$ `[1.1.1.2.2 ← 1.1.1]`
4. $f(f(x,f(y,z)),f(z,f(z,f(x,y)))) = x$ `[2.1.1 ← 1.1.1 (flip)]`
5. $f(f(x,f(f(y,z),x)),f(y,f(z,x))) = y$ `[2.1.1 → 1.1.1.1.2]`
6. $f(f(f(x,f(y,z)),x),f(y,f(z,x))) = y$ `[2.1.1 → 1.1.1.1]`
7. $f(f(f(x,y),z),f(y,f(y,f(z,x)))) = z$ `[4.1.1.1 ← 2.1.2]`
8. $f(f(x,f(y,z)),f(f(z,f(x,y)),z)) = x$ `[4.1.1.2 ← 2.1.2]`
9. $f(f(x,f(f(y,z),x)),f(z,f(y,x))) = z$ `[5.1.1.1.2.1 ← 2.1.2]`
10. $f(f(f(x,f(y,z)),x),f(y,f(x,z))) = y$ `[6.1.1.2.2 ← 2.1.2]`
11. $f(f(f(x,y),z),f(x,f(x,f(z,y)))) = z$ `[7.1.1.1.1 ← 2.1.2]`
12. $f(f(x,f(y,z)),f(f(y,f(x,z)),y)) = x$ `[8.1.1.1.2 ← 2.1.2]`
13. $f(f(x,f(f(y,z),x)),f(f(y,x),z)) = z$ `[9.1.1.2 ← 2.1.2]`
14. $f(f(x,y),f(x,f(f(f(x,y),f(f(x,y),z)),y))) = x$ `[10.1.1.1 ← 10.1.1]`
15. $f(f(f(x,y),z),f(x,f(f(z,y),x))) = z$ `[11.1.1.2.2 ← 2.1.2]`
16. $f(f(f(f(x,y),z),z),f(f(x,z),y)) = y$ `[13.1.1.1 ← 2.1.2]`
17. $f(f(f(x,y),z),f(x,f(f(y,z),x))) = z$ `[15.1.1.2.2.1 ← 2.1.2]`
18. $f(f(f(f(x,f(y,f(y,f(x,z)))),f(z,y)),f(z,y)),x) = f(y,f(y,f(x,z)))$ `[16.1.1.2 ← 4.1.1]`
19. $f(f(x,f(y,x)),f(x,y)) = x$ `[3.1.1 → 14.1.1.2.2]`
20. $f(f(x,f(x,y)),f(x,y)) = x$ `[19.1.1.1.2 ← 2.1.2]`
21. $f(f(x,f(y,x)),f(y,x)) = x$ `[19.1.1.2 ← 2.1.2]`
22. $f(f(f(x,y),y),f(x,y)) = y$ `[21.1.1.1 ← 2.1.2]`
23. $f(x,f(x,f(y,x))) = f(y,x)$ `[22.1.1.1 ← 21.1.1]`
24. $f(x,f(x,f(x,y))) = f(x,y)$ `[22.1.1.1 ← 20.1.1]`
25. $f(x,f(f(x,y),x)) = f(x,y)$ `[24.1.1.2 ← 2.1.2]`
26. $f(f(f(x,y),x),x) = f(x,y)$ `[25.1.1 ← 2.1.2]`
27. $f(f(x,y),f(f(x,y),f(f(x,x),f(x,y)))) = x$ `[26.1.1 → 17.1.1.1]`

* 28. $f(f(x,x),f(x,y)) \;=\; x$ `[27.1.1` $\leftarrow$ `23.1.1]` (Sheffer 1)

29. $f(f(f(f(f(f(x,f(y,z)),f(f(y,f(x,z)),y)),f(x,f(y,f(x,z)))),$
$f(x,f(y,f(x,z)))),f(x,f(y,z)))$
$\;=\; f(f(y,f(x,z)),f(f(y,f(x,z)),f(f(x,f(y,z)),x)))$
`[18.1.1.1.1.1.2.2` $\leftarrow$ `12.1.1]`

* 30. $f(f(x,x),f(y,x)) \;=\; x$ `[28.1.1.2` $\leftarrow$ `23.1.1]` (Sheffer 1)

* 31. $f(f(x,y),f(x,x)) \;=\; x$ `[28.1.1` $\leftarrow$ `2.1.2]` (Sheffer 1)

32. $f(f(x,x),f(y,f(y,f(x,z)))) \;=\; x$ `[30.1.1.2` $\leftarrow$ `18.1.1]`

33. $f(f(x,x),f(f(y,y),y)) \;=\; x$ `[32.1.1.2.2` $\leftarrow$ `32.1.1]`

34. $f(f(x,x),f(y,f(y,y))) \;=\; x$ `[33.1.1.2.1` $\leftarrow$ `30.1.1]`

* 35. $f(x,f(y,f(y,y))) \;=\; f(x,x)$ `[34.1.1.1` $\leftarrow$ `30.1.1]` (Sheffer 2)

36. $f(f(f(x,f(x,f(y,f(x,z)))),f(x,f(y,f(x,z)))),f(x,f(y,z)))$
$\;=\; f(f(y,f(x,z)),f(f(y,f(x,z)),f(f(x,f(y,z)),x)))$
`[29.1.1.1.1.1` $\leftarrow$ `12.1.1]`

37. $f(f(f(x,f(x,f(y,f(x,z)))),f(x,f(y,f(x,z)))),f(x,f(y,z)))$
$\;=\; f(f(y,f(x,z)),y)$ `[36.1.2.2` $\leftarrow$ `12.1.1]`

38. $f(f(x,f(y,z)),x) \;=\; f(y,f(y,f(x,z)))$ `[37.1.1.1` $\leftarrow$ `20.1.1 (flip)]`

39. $f(f(x,f(y,y)),x) \;=\; f(x,y)$ `[38.1.2` $\leftarrow$ `23.1.1]`

40. $f(x,f(y,f(y,f(x,z)))) \;=\; f(x,f(y,z))$ `[38.1.1` $\rightarrow$ `25.1.1.2]`

41. $f(x,f(x,f(y,y))) \;=\; f(x,y)$ `[39.1.1` $\leftarrow$ `2.1.2]`

42. $f(x,f(x,y)) \;=\; f(x,f(y,y))$ `[39.1.1` $\rightarrow$ `25.1.1.2]`

43. $f(x,f(x,f(y,y))) \;=\; f(y,x)$ `[41.1.2` $\leftarrow$ `2.1.2]`

44. $f(x,f(f(y,x),f(y,x))) \;=\; f(y,x)$ `[42.1.1` $\leftarrow$ `23.1.1 (flip)]`

45. $f(x,f(x,f(x,f(y,y)))) \;=\; f(x,f(x,y))$ `[42.1.2` $\leftarrow$ `24.1.2 (flip)]`

46. $f(x,f(y,f(y,f(x,f(z,z))))) \;=\; f(x,f(y,f(x,z)))$ `[40.1.1.2.2.2` $\leftarrow$ `42.1.1]`

47. $f(x,f(y,y)) \;=\; f(x,f(x,y))$ `[45.1.1` $\leftarrow$ `24.1.1]`

48. $f(x,f(y,f(z,z))) \;=\; f(x,f(y,f(x,z)))$ `[46.1.1` $\leftarrow$ `40.1.1]`

49. $f(x,f(y,f(x,z))) \;=\; f(x,f(y,f(z,z)))$ `[46.1.1` $\rightarrow$ `40.1.1]`

50. $f(x,f(y,f(x,f(z,z)))) \;=\; f(x,f(y,z))$ `[48.1.1.2.2` $\leftarrow$ `31.1.1 (flip)]`

51. $f(x,f(y,f(y,z))) \;=\; f(x,f(y,f(x,z)))$ `[48.1.1.2` $\leftarrow$ `47.1.1]`

52. $f(x,f(y,f(z,z))) \;=\; f(x,f(y,f(z,x)))$ `[48.1.2.2.2` $\leftarrow$ `2.1.2]`

53. $f(x,f(y,f(z,x))) \;=\; f(x,f(y,f(z,z)))$ `[49.1.1.2.2` $\leftarrow$ `2.1.2]`

54. $f(f(x,y),f(x,f(z,f(y,y)))) \;=\; f(f(x,y),f(x,f(z,z)))$
`[52.1.2.2` $\leftarrow$ `49.1.1 (flip)]`

55. $f(f(x,f(y,z)),f(x,f(y,z))) \;=\; f(f(x,f(y,z)),f(x,f(z,z)))$
`[54.1.1.2.2` $\leftarrow$ `44.1.1]`

56. $f(x,f(y,f(y,f(z,z)))) \;=\; f(x,f(y,z))$ `[50.1.1` $\leftarrow$ `51.1.2]`

57. $f(f(x,f(y,f(x,f(z,z)))),f(x,f(z,z))) = f(f(x,f(y,z)),f(x,f(y,z)))$
`[50.1.2 → 55.1.2.1 (flip)]`

58. $f(f(x,f(y,y)),f(x,f(z,f(x,f(y,y))))) = f(f(x,f(z,y)),f(x,f(z,y)))$
`[57.1.1 ← 2.1.2]`

59. $f(f(x,f(y,y)),f(x,f(z,z))) = f(f(x,f(z,y)),f(x,f(z,y)))$
`[58.1.1 ← 53.1.1]`

60. $f(x,f(y,z)) = f(x,f(z,y))$ `[56.1.1.2 ← 43.1.1]`

61. $f(f(x,y),z) = f(z,f(y,x))$ `[60.1.1 ← 2.1.2]`

62. $f(f(x,y),f(z,u)) = f(f(u,z),f(y,x))$ `[61.1.1 ← 60.1.2]`

* 63. $f(f(f(x,x),y),f(f(z,z),y)) = f(f(y,f(x,z)),f(y,f(x,z)))$
`[62.1.2 ← 59.1.1]` (flip of Sheffer 3)

Theorem 2. The pair consisting of Commutativity and *any one* of the axioms in { 8, 11, 13, 15, 17, 19, 21, 23 } forms a basis.

In each case, we either derive Axiom 7—the result then follows from Theorem 1—or derive one of the known systems directly.

Theorem 3. Axioms 9 and 11 together form a basis.

Proof. It suffices to derive the axioms of the Sheffer system. These axioms are flagged with a * symbol in the following proof.

1. $f(f(x,f(f(y,x),x)),f(y,f(x,z))) = y$ (Axiom 9)
2. $f(f(x,f(f(y,z),x)),f(y,f(z,x))) = y$ (Axiom 11)
3. $f(f(x,y),f(x,f(y,f(z,f(f(f(x,y),z),z))))) = x$ `[2.1.1.1 ← 1.1.1]`
4. $f(f(x,f(f(f(y,f(f(x,z),y)),x),x)),x) = f(y,f(f(x,z),y))$
`[2.1.1 → 1.1.1.2]`
5. $f(f(f(x,f(f(y,x),x)),y),y) = f(x,f(f(y,x),x))$ `[3.1.1.2 ← 1.1.1]`
6. $f(f(f(x,y),f(f(z,f(x,y)),f(x,y))),f(z,x)) = z$ `[3.1.1 → 1.1.1.2.2]`
7. $f(f(x,f(f(f(x,y),x),x)),x) = f(x,y)$ `[3.1.1 → 1.1.1.2]`
8. $f(f(f(x,y),f(f(x,f(x,y)),f(x,y))),f(x,y))$
$= f(f(x,y),f(x,f(y,f(z,f(f(f(x,y),z),z)))))$ `[7.1.1.1.2.1.1 ← 3.1.1]`
9. $f(f(x,f(f(x,y),x)),x) = f(x,f(f(f(x,y),x),x))$ `[7.1.1.1.2.1 ← 7.1.1]`
10. $f(f(f(x,f(f(x,y),x)),x),x) = f(x,y)$ `[9.1.2 → 7.1.1.1]`
11. $f(f(x,f(x,y)),x) = f(x,f(f(x,y),x))$ `[10.1.1 → 7.1.1.1.2]`
12. $f(f(x,f(y,f(f(x,y),y))),x) = f(y,f(f(x,y),y))$ `[4.1.1.1.2 ← 5.1.1]`
13. $f(f(f(f(x,y),f(x,f(f(f(x,y),x),x))),f(x,y)),x) = f(x,y)$
`[12.1.2 → 7.1.1.1]`
14. $f(f(f(x,y),f(f(x,f(x,y)),f(x,y))),f(x,y)) = x$ `[8.1.2 ← 3.1.1]`

15. $f(f(f(x,f(f(f(x,y),x),x)),f(f(x,f(f(x,f(f(x,y),x)),x)),$
$f(x,f(f(f(x,y),x),x)))),f(x,f(f(f(x,y),x),x)))$
$= x$ [14.1.1.1.2.1.2 $\leftarrow$ 9.1.2]

16. $f(f(f(x,f(f(f(x,y),x),x)),x),f(x,f(f(f(x,y),x),x))) = x$
[15.1.1.1.2 $\leftarrow$ 2.1.1]

17. $f(f(x,y),f(x,f(f(f(x,y),x),x))) = x$ [16.1.1.1 $\leftarrow$ 7.1.1]

18. $f(f(x,f(x,y)),x) = f(x,y)$ [17.1.1 $\rightarrow$ 13.1.1.1.1]

19. $f(x,f(f(x,y),x)) = f(x,y)$ [18.1.1 $\leftarrow$ 11.1.1]

20. $f(f(f(x,y),x),x) = f(x,y)$ [19.1.1 $\rightarrow$ 10.1.1.1.1]

21. $f(f(x,y),f(x,f(y,x))) = x$ [19.1.1 $\rightarrow$ 2.1.1.1]

22. $f(f(f(f(x,y),x),f(f(x,y),f(f(x,y),x))),f(x,f(f(f(x,y),x),z))) = x$
[19.1.1 $\rightarrow$ 1.1.1.1.2.1]

23. $f(f(x,y),f(x,f(x,y))) = x$ [20.1.1 $\rightarrow$ 17.1.1.2.2]

24. $f(f(x,f(x,y)),f(x,y)) = x$ [21.1.1.2 $\leftarrow$ 19.1.1]

25. $f(f(f(x,y),x),f(x,y)) = x$ [24.1.1 $\rightarrow$ 14.1.1.1.2]

26. $f(f(f(x,y),x),f(x,x)) = x$ [24.1.1 $\rightarrow$ 6.1.1.1.2]

27. $f(f(f(x,y),x),f(x,f(f(x,y),z))) = x$ [24.1.1 $\rightarrow$ 1.1.1.1.2]

* 28. $f(f(x,y),f(x,x)) = x$ [26.1.1.1 $\leftarrow$ 18.1.1] (Sheffer 1)

29. $f(f(x,f(f(y,z),x)),f(y,y)) = y$ [26.1.1.1 $\leftarrow$ 4.1.1]

30. $f(f(f(x,x),x),f(y,y)) = y$ [29.1.1.1.2 $\leftarrow$ 29.1.1]

31. $f(f(f(x,f(y,z)),x),f(z,z)) = z$ [29.1.1.1.2 $\leftarrow$ 2.1.1]

32. $f(f(x,f(x,x)),f(y,y)) = y$ [30.1.1.1.1 $\leftarrow$ 28.1.1]

33. $f(f(f(f(x,f(x,x)),f(y,y)),f(x,f(x,x))),f(x,f(x,x))) = y$
[32.1.1 $\leftarrow$ 20.1.2]

34. $f(f(f(x,f(x,x)),f(f(f(f(x,f(x,x)),f(y,y)),f(x,f(x,x))),f(x,f(x,x)))),$
$f(x,f(x,x))) = y$ [32.1.1 $\leftarrow$ 7.1.2]

* 35. $f(f(x,y),f(y,y)) = y$ [31.1.1.1 $\leftarrow$ 30.1.1] (Sheffer 1)

* 36. $f(f(x,x),f(y,x)) = x$ [31.1.1 $\leftarrow$ 20.1.1] (Sheffer 1)

37. $f(f(f(f(x,f(y,z)),x),f(x,f(y,z))),f(f(x,f(y,z)),z)) = f(x,f(y,z))$
[27.1.1.2.2 $\leftarrow$ 31.1.1]

38. $f(f(x,y),f(x,f(f(f(x,y),x),z))) = x$ [22.1.1.1 $\leftarrow$ 23.1.1]

39. $f(f(f(x,f(x,x)),y),f(x,f(x,x))) = y$ [34.1.1.1.2 $\leftarrow$ 33.1.1]

* 40. $f(x,f(y,f(y,y))) = f(x,x)$ [39.1.1.1 $\leftarrow$ 32.1.1] (Sheffer 2)

41. $f(x,f(f(x,f(y,z)),z)) = f(x,f(y,z))$ [25.1.1 $\rightarrow$ 37.1.1.1]

42. $f(f(x,f(y,z)),f(x,z)) = x$ [38.1.1.2.2 $\leftarrow$ 31.1.1]

43. $f(f(x,y),f(x,f(y,z))) = x$ [42.1.1.1.2 $\leftarrow$ 42.1.1]

44. $f(x,f(f(y,x),x)) = f(y,x)$ [42.1.1.1 $\leftarrow$ 35.1.1]

45. $f(f(x,y),f(x,f(z,y))) \;=\; x$ [43.1.1.2.2 ← 44.1.1]

46. $f(f(f(x,y),f(z,y)),x) \;=\; f(x,y)$ [45.1.1.2 ← 45.1.1]

47. $f(x,y) \;=\; f(y,x)$ [46.1.1.1 ← 35.1.1]

48. $f(x,f(x,f(y,x))) \;=\; f(y,x)$ [47.1.1 → 44.1.1.2]

49. $f(f(x,f(x,f(y,z))),f(y,f(z,x))) \;=\; y$ [47.1.1 → 2.1.1.1.2]

50. $f(f(f(x,f(y,z)),x),f(y,f(z,x))) \;=\; y$ [49.1.1.1 ← 47.1.2]

51. $f(f(f(x,f(y,z)),x),f(y,f(x,z))) \;=\; y$ [50.1.1.2.2 ← 47.1.2]

52. $f(f(f(x,f(y,z)),f(f(f(f(y,f(x,z)),f(f(f(x,f(y,z)),x),f(y,f(x,z)))),$
$f(x,f(y,z))),f(x,f(y,z)))),f(x,f(y,z)))$
$= \; f(f(y,f(x,z)),y)$ [51.1.1 → 4.1.2.2]

53. $f(f(f(x,f(y,z)),f(f(f(f(y,f(x,z)),y),f(x,f(y,z))),f(x,f(y,z)))),$
$f(x,f(y,z))) \;=\; f(f(y,f(x,z)),y)$ [52.1.1.1.2.1.1.2 ← 51.1.1]

54. $f(f(f(x,f(y,z)),f(x,f(x,f(y,z)))),f(x,f(y,z))) \;=\; f(f(y,f(x,z)),y)$
[53.1.1.1.2.1 ← 51.1.1]

55. $f(f(x,f(y,z)),x) \;=\; f(y,f(y,f(x,z)))$ [54.1.1.1 ← 45.1.1 (flip)]

56. $f(x,f(y,f(y,f(x,z)))) \;=\; f(x,f(y,z))$ [55.1.1 → 19.1.1.2]

57. $f(x,f(x,y)) \;=\; f(x,f(y,y))$ [56.1.1.2 ← 48.1.1]

58. $f(x,f(y,f(y,f(x,z)))) \;=\; f(x,f(z,y))$ [56.1.2.2 ← 47.1.2]

59. $f(x,f(y,f(y,f(x,f(z,z))))) \;=\; f(x,f(y,f(x,z)))$ [57.1.1 → 56.1.1.2.2.2]

60. $f(x,f(y,z)) \;=\; f(x,f(z,y))$ [58.1.1 ← 56.1.1]

61. $f(f(x,y),z) \;=\; f(z,f(y,x))$ [60.1.1 ← 47.1.2]

62. $f(x,f(y,f(z,z))) \;=\; f(x,f(y,f(x,z)))$ [59.1.1 ← 56.1.1]

63. $f(x,f(y,f(x,f(z,z)))) \;=\; f(x,f(y,z))$ [62.1.1.2.2 ← 36.1.1 (flip)]

64. $f(x,f(y,f(z,z))) \;=\; f(x,f(y,f(z,x)))$ [62.1.2.2.2 ← 47.1.2]

65. $f(f(x,f(y,z)),f(x,f(z,z))) \;=\; f(f(x,f(y,z)),f(x,f(y,z)))$
[62.1.2.2 ← 41.1.1]

66. $f(f(x,f(y,f(x,f(z,z)))),f(x,f(z,z))) \;=\; f(f(x,f(y,z)),f(x,f(y,z)))$
[65.1.1.1 ← 63.1.2]

67. $f(f(x,f(y,y)),f(x,f(z,f(x,f(y,y))))) \;=\; f(f(x,f(z,y)),f(x,f(z,y)))$
[66.1.1 ← 47.1.2]

68. $f(f(x,f(y,y)),f(x,f(z,z))) \;=\; f(f(x,f(z,y)),f(x,f(z,y)))$
[67.1.1 ← 64.1.2]

69. $f(f(x,y),f(z,u)) \;=\; f(f(u,z),f(y,x))$ [61.1.1 ← 60.1.2]

* 70. $f(f(f(x,x),y),f(f(z,z),y)) \;=\; f(f(y,f(x,z)),f(y,f(x,z)))$
[69.1.2 ← 68.1.1] (flip of Sheffer 3)

We conjecture that several of the pairs consisting of one axiom from { 7, 8, 11, 13, 15, 17, 19, 21, 23 } and one axiom from { 1, 2, 3, 4, 5, 6, 9, 10, 12, 14, 16, 18, 20, 22, 24, 25, 26a, 27a } suffice. The following two theorems are representative of the results we have obtained to date.

Theorem 4. The pair consisting of Axiom 1 and *any one* of the axioms in { 7, 8, 11, 13, 15, 17, 19, 21, 23 } forms a basis.

Theorem 5. The pair consisting of Axiom 17 and *any one* of the axioms in { 1, 2, 3, 4, 5, 6, 9, 10, 12, 14, 16, 17, 18, 20, 22, 25, 26a, 27a } forms a basis.

In each case, we derive Commutativity. The result then follows from Theorem 2.

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