Short 2-Bases for Boolean Algebra in Terms of the Sheffer Stroke

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June 26, 2000

Abstract

In this article, we summarize several new results concerning short 2-bases for Boolean algebra in terms of the Sheffer stroke. Two especially short bases are presented, each consisting of a pair of equations having a total of only six applications of the Sheffer stroke operator. Our proofs of correctness of these new results relied heavily on the use of an automated reasoning program.

1 Introduction

In [3], we presented several new 2-axiom equational bases (2-bases) for Boolean algebra in terms of the Sheffer stroke '|' (where $x|y = x' \wedge y'$). In this report, we summarize results from a continuation of that work, including several new 2-bases.

2 Summary of Previous Results

In this section, we summarize the theorems presented in [3]. The axiom names in these theorems refer to the following list of equations, in which $f(x, y)$ denotes $x|y$, an application of the Sheffer stroke.

1. $f(f(x, f(x, f(x, y))), f(y, f(x, z))) = y$
2. $f(f(x, f(x, f(x, y))), f(y, f(z, x))) = y$
3. $f(f(x, f(x, f(y, x))), f(y, f(x, z))) = y$
4. $f(f(x, f(x, f(y, x))), f(y, f(z, x))) = y$
5. $f(f(x, f(x, f(y, y))), f(y, f(x, z))) = y$
6. $f(f(x, f(x, f(y, y))), f(y, f(z, x))) = y$

*This work was partially supported by National Science Foundation grant no. CDA-9503064.

7. $f(f(x,f(x,f(y,z))),f(y,f(z,x))) = y$
8. $f(f(x,f(x,f(y,z))),f(z,f(x,y))) = z$
9. $f(f(x,f(f(y,x),x)),f(y,f(x,z))) = y$
10. $f(f(x,f(f(y,x),x)),f(y,f(z,x))) = y$
11. $f(f(x,f(f(y,z),x)),f(y,f(z,x))) = y$
12. $f(f(f(x,y),z),f(x,f(x,f(z,x)))) = z$
13. $f(f(f(x,y),z),f(x,f(x,f(z,y)))) = z$
14. $f(f(f(x,y),z),f(x,f(f(x,z),x))) = z$
15. $f(f(f(x,y),z),f(x,f(f(y,z),x))) = z$
16. $f(f(f(x,y),z),f(x,f(f(z,x),x))) = z$
17. $f(f(f(x,y),z),f(x,f(f(z,y),x))) = z$
18. $f(f(f(x,y),z),f(x,f(f(z,z),x))) = z$
19. $f(f(f(x,y),z),f(y,f(y,f(z,x)))) = z$
20. $f(f(f(x,y),z),f(y,f(y,f(z,y)))) = z$
21. $f(f(f(x,y),z),f(y,f(f(x,z),y))) = z$
22. $f(f(f(x,y),z),f(y,f(f(y,z),y))) = z$
23. $f(f(f(x,y),z),f(y,f(f(z,x),y))) = z$
24. $f(f(f(x,y),z),f(y,f(f(z,y),y))) = z$
25. $f(f(f(x,y),z),f(y,f(f(z,z),y))) = z$

26a. $f(f(x,y),f(x,f(y,z))) = x$
26b. $f(x,y) = f(y,x)$ (Commutativity)

27a. $f(f(x,z),f(x,f(y,z))) = x$
27b. $f(x,y) = f(y,x)$ (Commutativity)

These equations were brought to our attention by Dana Scott. They originated from an investigation in progress by Steven Wolfram and David Hillman.

Theorem 1. [3] Axiom 7 and Commutativity together form a basis.

Theorem 2. [3] The pair consisting of Commutativity and *any one* of the axioms in { 8, 11, 13, 15, 17, 19, 21, 23 } forms a basis.

Theorem 3. [3] Axioms 9 and 11 together form a basis.

Theorem 4. [3] The pair consisting of Axiom 1 and *any one* of the axioms in { 7, 8, 11, 13, 15, 17, 19, 21, 23 } forms a basis.

Theorem 5. [3] The pair consisting of Axiom 17 and *any one* of the axioms in { 1, 2, 3, 4, 5, 6, 9, 10, 12, 14, 16, 17, 18, 20, 22, 25, 26a, 27a } forms a basis.

3 New Results

Since the original report, we have identified several new 2-bases for Boolean algebra in terms of the Sheffer stroke. It is perhaps especially notable that {26a, Commutativity} and {27a, Commutativity} are 2-bases, since they are relatively simple systems. All of our correctness proofs relied heavily on the automated reasoning program Otter [1].

Theorem 6. The pair consisting of Axiom 26a and Commutativity forms a basis.

Proof. It suffices to derive the axioms of the original Sheffer system [2].

$$f(f(x,x),f(x,x)) = x \qquad \text{(Sheffer 1)}$$
$$f(x,f(y,f(y,y))) = f(x,x) \qquad \text{(Sheffer 2)}$$
$$f(f(x,f(y,z)),f(x,f(y,z))) = f(f(f(y,y),x),f(f(z,z),x)) \qquad \text{(Sheffer 3)}$$

These axioms are flagged with a * symbol in the following proof.

1. $f(f(x,y),f(x,f(y,z))) = x$ `(Axiom 26a)`
2. $f(x,y) = f(y,x)$ `(Commutativity)`
3. $f(f(x,f(y,z)),f(x,y)) = x$ `[1.1.1.2.2 ← 1.1.1]`
4. $f(f(f(x,y),x),x) = f(x,y)$ `[1.1.1.2 ← 1.1.1]`
5. $f(f(x,y),f(x,f(z,y))) = x$ `[2.1.1 → 1.1.1.2.2]`
6. $f(f(x,y),f(y,f(x,z))) = y$ `[2.1.1 → 1.1.1.1]`
7. $f(f(x,f(y,z)),f(x,z)) = x$ `[3.1.1.1.2 ← 2.1.2]`
8. $f(f(x,f(y,z)),f(y,x)) = x$ `[3.1.1.2 ← 2.1.2]`
9. $f(x,f(f(x,y),f(z,f(x,f(u,y))))) = f(x,y)$ `[5.1.1.1 ← 5.1.1]`
10. $f(f(x,y),f(y,f(z,x))) = y$ `[5.1.1.1 ← 2.1.2]`
11. $f(f(f(x,y),f(z,y)),x) = f(x,y)$ `[5.1.1.2 ← 5.1.1]`
12. $f(f(f(x,y),z),f(z,y)) = z$ `[6.1.1.2.2 ← 6.1.1]`
13. $f(f(x,y),f(f(x,z),y)) = y$ `[6.1.1.2 ← 2.1.2]`
14. $f(f(x,f(y,z)),f(z,x)) = x$ `[7.1.1.2 ← 2.1.2]`
15. $f(f(f(x,f(y,z)),z),x) = f(x,f(y,z))$ `[7.1.1 → 5.1.1.2]`
16. $f(x,f(y,f(x,f(y,z)))) = f(x,f(y,z))$ `[8.1.1.1 ← 8.1.1]`
17. $f(f(f(x,y),f(x,z)),z) = f(x,z)$ `[10.1.1.2 ← 6.1.1]`
18. $f(x,f(f(y,x),f(z,y))) = f(y,x)$ `[10.1.1 → 7.1.1.1]`

* 19. $f(f(x,y),f(x,x)) = x$ `[12.1.1.1 ← 4.1.1]` (generalization of Sheffer 1)

20. $f(f(f(x,y),z),f(y,z)) = z$ `[12.1.1.2 ← 2.1.2]`

* 21. $f(f(x,y),f(y,y)) = y$ `[19.1.1.1 ← 2.1.2]` (generalization of Sheffer 1)

22. $f(x,f(f(x,y),f(z,f(x,x)))) = f(x,y)$ `[19.1.1 → 5.1.1.1]`
23. $f(x,f(y,f(x,f(z,y)))) = f(x,f(z,y))$ `[14.1.1.1 ← 7.1.1]`

24. $f(x,f(y,f(x,x))) \ = \ f(x,x)$ `[20.1.1.1 ← 19.1.1]`

25. $f(f(x,f(y,z)),f(x,f(u,f(y,x)))) \ = \ f(f(y,x),f(x,f(y,z)))$ `[18.1.1.2.1 ← 6.1.1]`

26. $f(x,f(y,f(x,y))) \ = \ f(x,x)$ `[15.1.1.1 ← 3.1.1 (flip)]`

27. $f(f(x,x),f(f(x,y),f(z,f(x,z)))) \ = \ f(z,f(x,z))$ `[26.1.1 → 13.1.1.1]`

28. $f(x,f(y,f(x,f(z,y)))) \ = \ f(x,f(y,z))$ `[16.1.1.2.2.2 ← 2.1.2]`

29. $f(x,f(y,y)) \ = \ f(x,f(y,x))$ `[16.1.1.2 ← 26.1.1]`

30. $f(f(x,x),y) \ = \ f(y,f(x,y))$ `[29.1.1 ← 2.1.2]`

31. $f(x,f(y,y)) \ = \ f(x,f(x,y))$ `[29.1.2.2 ← 2.1.2]`

32. $f(f(x,y),y) \ = \ f(y,f(x,x))$ `[29.1.2 ← 2.1.2 (flip)]`

33. $f(x,f(y,x)) \ = \ f(f(y,y),x)$ `[29.1.1 → 2.1.1]`

34. $f(f(f(x,y),f(x,y)),f(x,x)) \ = \ f(f(x,x),x)$ `[30.1.2.2 ← 19.1.1]`

35. $f(f(x,x),y) \ = \ f(y,f(y,x))$ `[30.1.2.2 ← 2.1.2]`

36. $f(x,f(x,f(y,y))) \ = \ f(x,y)$ `[31.1.1.2 ← 19.1.1 (flip)]`

37. $f(f(x,y),x) \ = \ f(x,f(y,y))$ `[31.1.2 ← 2.1.2 (flip)]`

38. $f(x,f(x,f(y,y))) \ = \ f(y,x)$ `[35.1.1.1 ← 19.1.1 (flip)]`

39. $f(x,f(x,y)) \ = \ f(x,f(y,y))$ `[35.1.1 → 2.1.1]`

40. $f(f(f(x,y),f(z,y)),f(x,x)) \ = \ f(f(x,y),f(f(x,y),f(z,y)))$ `[37.1.1.1 ← 11.1.1 (flip)]`

41. $f(x,f(f(x,y),f(z,f(x,z)))) \ = \ f(x,y)$ `[22.1.1.2.2 ← 29.1.1]`

42. $f(x,f(y,z)) \ = \ f(x,f(z,y))$ `[28.1.1 ← 23.1.1]`

43. $f(f(x,y),z) \ = \ f(z,f(y,x))$ `[42.1.1 ← 2.1.2]`

44. $f(x,f(y,z)) \ = \ f(f(z,y),x)$ `[42.1.2 ← 2.1.2]`

45. $f(f(x,y),f(z,u)) \ = \ f(f(u,z),f(y,x))$ `[43.1.1 ← 42.1.2]`

46. $f(f(x,y),f(x,f(x,f(z,y)))) \ = \ f(f(x,y),f(x,y))$ `[9.1.1 → 16.1.1.2 (flip)]`

47. $f(f(f(x,y),f(x,y)),f(x,x)) \ = \ f(x,f(x,x))$ `[34.1.2 ← 44.1.2]`

48. $f(f(f(x,x),f(x,x)),f(f(x,y),f(z,f(x,z))))$
$\quad = \ f(f(f(x,y),f(z,f(x,z))),f(z,f(x,z)))$ `[27.1.1 → 33.1.1.2 (flip)]`

49. $f(f(x,f(y,f(z,y))),f(x,f(x,f(z,z))))$
$\quad = \ f(f(x,f(y,f(z,y))),f(x,f(y,f(z,y))))$ `[46.1.1.2.2.2 ← 26.1.1]`

50. $f(f(x,f(y,z)),f(x,f(u,f(y,x)))) \ = \ x$ `[25.1.2 ← 6.1.1]`

51. $f(x,f(y,f(x,f(z,f(y,x))))) \ = \ f(x,x)$ `[50.1.1 → 15.1.1.1 (flip)]`

52. $f(x,f(y,f(z,f(x,y)))) \ = \ f(x,f(y,y))$ `[51.1.1 → 16.1.1.2 (flip)]`

53. $f(x,f(f(y,y),f(z,f(x,f(x,y))))) \ = \ f(x,f(f(y,y),f(y,y)))$ `[52.1.1.2.2.2 ← 39.1.2]`

54. $f(x,f(y,f(f(y,x),z))) \ = \ f(x,f(y,y))$ `[52.1.1.2.2 ← 44.1.1]`

55. $f(f(x,y),f(f(x,y),f(x,y))) \ = \ f(x,f(x,x))$ `[40.1.1 ← 47.1.1 (flip)]`

56. $f(f(x,f(y,y)),f(f(f(y,x),x),f(f(y,x),x)))$
$= f(f(y,x),f(f(y,x),f(y,x)))$ [55.1.1.1 ← 32.1.1]

57. $f(x,f(f(y,y),f(z,f(x,f(x,y))))) = f(x,y)$ [53.1.2.2 ← 19.1.1]

58. $f(f(x,f(y,y)),f(f(x,f(y,y)),f(f(y,x),x)))$
$= f(f(y,x),f(f(y,x),f(y,x)))$ [56.1.1.2.1 ← 32.1.1]

59. $f(f(x,f(y,y)),f(f(x,f(y,y)),f(x,f(y,y))))$
$= f(f(y,x),f(f(y,x),f(y,x)))$ [58.1.1.2.2 ← 32.1.1]

60. $f(f(x,y),f(f(x,y),f(x,y))) = f(y,f(y,y))$ [59.1.1 ← 55.1.1 (flip)]

61. $f(x,f(x,x)) = f(y,f(y,y))$ [60.1.1 ← 55.1.1]

62. $f(f(x,f(y,f(z,y))),f(x,f(y,f(z,y)))) = f(f(x,f(y,f(z,y))),f(x,z))$
[49.1.1.2 ← 36.1.1 (flip)]

63. $f(f(f(x,x),f(x,x)),f(f(x,y),f(z,f(x,z))))$
$= f(f(z,f(x,z)),f(f(x,y),f(x,y)))$ [48.1.2 ← 32.1.1]

64. $f(f(x,f(y,x)),f(f(y,z),f(y,z))) = f(y,f(f(y,z),f(x,f(y,x))))$
[63.1.1.1 ← 19.1.1 (flip)]

65. $f(f(x,f(y,x)),f(f(y,z),f(y,z))) = f(y,z)$ [64.1.2 ← 41.1.1]

* 66. $f(x,f(y,f(y,y))) = f(x,x)$ [61.1.1 → 22.1.1.2] (Sheffer 2)

67. $f(f(x,y),f(x,f(z,f(z,z)))) = x$ [61.1.1 → 1.1.1.2.2]

68. $f(f(x,f(f(y,z),x)),f(y,f(f(y,z),f(y,f(u,f(u,u))))))$
$= f(f(y,z),f(y,f(u,f(u,u))))$ [67.1.1 → 65.1.1.2.1]

69. $f(f(x,f(f(y,z),x)),f(y,y)) = f(f(y,z),f(y,f(u,f(u,u))))$
[68.1.1.2.2 ← 67.1.1]

70. $f(f(x,f(f(y,z),x)),f(y,y)) = y$ [69.1.2 ← 67.1.1]

71. $f(f(x,f(x,f(y,z))),f(z,z)) = z$ [70.1.1.1.2 ← 44.1.2]

72. $f(x,f(f(y,f(y,f(z,x))),x)) = f(y,f(y,f(z,x)))$ [71.1.1 → 7.1.1.1]

73. $f(x,f(y,f(z,f(z,f(u,f(y,x)))))) = f(x,f(y,y))$ [72.1.1 → 54.1.1.2.2]

74. $f(x,f(y,f(y,f(z,f(x,y))))) = f(x,f(y,f(x,x)))$ [73.1.1 → 16.1.1.2 (flip)]

75. $f(x,f(y,f(y,f(z,f(x,y))))) = f(x,x)$ [74.1.2 ← 24.1.1]

76. $f(f(x,x),f(y,f(y,f(z,f(y,f(x,y)))))) = f(f(x,x),f(x,x))$
[75.1.1.2.2.2.2 ← 33.1.2]

77. $f(f(x,x),f(y,f(y,f(z,f(y,f(x,y)))))) = x$ [76.1.2 ← 21.1.1]

78. $f(x,f(f(f(y,f(x,f(z,x))),f(y,f(x,f(z,x)))),z))$
$= f(x,f(y,f(x,f(z,x))))$ [77.1.1 → 57.1.1.2.2]

79. $f(x,f(f(f(y,f(x,f(z,x))),f(y,z)),z)) = f(x,f(y,f(x,f(z,x))))$
[78.1.1.2.1 ← 62.1.1]

80. $f(x,f(y,f(x,f(z,x)))) = f(x,f(y,z))$ [79.1.1.2 ← 17.1.1 (flip)]

81. $f(x,f(y,f(x,f(x,z)))) = f(x,f(y,z))$ [80.1.1.2.2.2 ← 2.1.2]

82. $f(x,f(y,f(x,f(z,z)))) = f(x,f(y,z))$ [80.1.1.2.2 ← 29.1.2]

83. $f(x, f(y, f(z, x))) = f(x, f(y, f(z, z)))$ `[81.1.1.2.2 ← 38.1.1]`

84. $f(f(x, f(y, z)), f(x, f(y, z))) = f(f(x, f(y, z)), f(x, f(z, z)))$
`[83.1.1.2 ← 23.1.1]`

85. $f(f(x, f(y, f(x, f(z, z)))), f(x, f(z, z))) = f(f(x, f(y, z)), f(x, f(y, z)))$
`[84.1.2.1 ← 82.1.2 (flip)]`

86. $f(f(x, f(y, y)), f(x, f(z, f(x, f(y, y))))) = f(f(x, f(z, y)), f(x, f(z, y)))$
`[85.1.1 ← 2.1.2]`

87. $f(f(x, f(y, y)), f(x, f(z, z))) = f(f(x, f(z, y)), f(x, f(z, y)))$
`[86.1.1 ← 83.1.1]`

* 88. $f(f(f(x, x), y), f(f(z, z), y)) = f(f(y, f(x, z)), f(y, f(x, z)))$
`[87.1.1 ← 45.1.2]` (flip of Sheffer 3)

Theorem 7. The pair consisting of Axiom 27a and Commutativity forms a basis.

Proof. It suffices to derive the axioms of the original Sheffer system. These axioms are flagged with a * symbol in the following proof.

1. $f(f(x, z), f(x, f(y, z))) = x$ `(Axiom 27a)`
2. $f(x, y) = f(y, x)$ `(Commutativity)`
3. $f(x, f(f(x, y), f(z, f(x, f(u, y))))) = f(x, y)$ `[1.1.1.1 ← 1.1.1]`
4. $f(f(f(x, y), f(z, y)), x) = f(x, y)$ `[1.1.1.2 ← 1.1.1]`
5. $f(f(x, f(y, z)), f(x, z)) = x$ `[2.1.1 ← 1.1.1 (flip)]`
6. $f(f(x, y), f(x, f(y, z))) = x$ `[2.1.1 → 1.1.1.2.2]`
7. $f(f(x, y), f(y, f(z, x))) = y$ `[2.1.1 → 1.1.1.1]`
8. $f(f(x, f(y, z)), f(x, y)) = x$ `[5.1.1.1.2 ← 2.1.2]`
9. $f(f(f(x, y), z), f(z, y)) = z$ `[5.1.1.1 ← 2.1.2]`
10. $f(f(x, f(y, z)), f(z, x)) = x$ `[5.1.1.2 ← 2.1.2]`
11. $f(f(f(x, f(y, z)), z), x) = f(x, f(y, z))$ `[5.1.1 → 1.1.1.2]`
12. $f(f(x, y), f(y, f(x, z))) = y$ `[6.1.1.1 ← 2.1.2]`
13. $f(f(f(x, y), x), x) = f(x, y)$ `[6.1.1.2 ← 6.1.1]`
14. $f(f(x, f(y, f(z, x))), y) = f(y, f(z, x))$ `[7.1.1.2 ← 5.1.1]`
15. $f(x, f(f(y, x), f(z, y))) = f(y, x)$ `[7.1.1 → 5.1.1.1]`
16. $f(f(x, f(y, z)), f(y, x)) = x$ `[8.1.1.2 ← 2.1.2]`
17. $f(f(f(x, y), z), f(y, z)) = z$ `[9.1.1.2 ← 2.1.2]`
18. $f(x, f(y, f(x, f(y, z)))) = f(x, f(y, z))$ `[10.1.1.1 ← 8.1.1]`
19. $f(x, f(y, f(x, f(z, y)))) = f(x, f(z, y))$ `[10.1.1.1 ← 5.1.1]`
20. $f(f(x, f(x, y)), x) = f(x, y)$ `[12.1.1.2 ← 6.1.1]`
21. $f(f(x, y), f(f(x, z), y)) = y$ `[12.1.1.2 ← 2.1.2]`

22. $f(f(f(x,y),f(x,z)),z) \ = \ f(x,z)$ [12.1.1 → 7.1.1.2]

* 23. $f(f(x,y),f(x,x)) \ = \ x$ [13.1.1 → 9.1.1.1] (generalization of Sheffer 1)

* 24. $f(f(x,y),f(y,y)) \ = \ y$ [23.1.1.1 ← 2.1.2] (generalization of Sheffer 1)

25. $f(x,f(f(x,y),f(z,f(x,x)))) \ = \ f(x,y)$ [23.1.1 → 1.1.1.1]

26. $f(x,f(y,f(x,x))) \ = \ f(x,x)$ [17.1.1.1 ← 23.1.1]

27. $f(f(x,f(y,z)),f(x,f(u,f(y,x)))) \ = \ f(f(y,x),f(x,f(y,z)))$
[15.1.1.2.1 ← 12.1.1]

28. $f(x,f(y,f(y,x))) \ = \ f(x,x)$ [11.1.1.1 ← 16.1.1 (flip)]

29. $f(x,f(y,f(x,y))) \ = \ f(x,x)$ [11.1.1.1 ← 8.1.1 (flip)]

30. $f(f(x,x),f(f(x,y),f(z,f(x,z)))) \ = \ f(z,f(x,z))$ [29.1.1 → 21.1.1.1]

31. $f(f(x,x),y) \ = \ f(y,f(y,x))$ [14.1.1.1 ← 28.1.1]

32. $f(x,f(x,f(y,y))) \ = \ f(y,x)$ [31.1.1.1 ← 23.1.1 (flip)]

33. $f(x,f(y,y)) \ = \ f(x,f(x,y))$ [31.1.1 ← 2.1.2]

34. $f(f(x,x),y) \ = \ f(y,f(x,y))$ [31.1.2.2 ← 2.1.2]

35. $f(x,f(x,y)) \ = \ f(x,f(y,y))$ [31.1.1 → 2.1.1]

36. $f(x,f(x,y)) \ = \ f(f(y,y),x)$ [32.1.1.2.2 ← 23.1.1]

37. $f(x,f(x,f(y,y))) \ = \ f(x,y)$ [32.1.2 ← 2.1.2]

38. $f(f(x,y),y) \ = \ f(y,f(x,x))$ [32.1.1 → 20.1.1.1]

39. $f(x,f(y,y)) \ = \ f(x,f(y,x))$ [33.1.2.2 ← 2.1.2]

40. $f(f(x,y),x) \ = \ f(x,f(y,y))$ [33.1.2 ← 2.1.2 (flip)]

41. $f(f(f(x,y),f(x,y)),f(x,x)) \ = \ f(f(x,x),x)$ [34.1.2.2 ← 23.1.1]

42. $f(x,f(y,x)) \ = \ f(f(y,y),x)$ [36.1.1.2 ← 2.1.2]

43. $f(f(f(x,y),f(z,y)),f(x,x)) \ = \ f(f(x,y),f(f(x,y),f(z,y)))$
[40.1.1.1 ← 4.1.1 (flip)]

44. $f(x,f(y,f(x,f(z,y)))) \ = \ f(x,f(y,z))$ [18.1.1.2.2.2 ← 2.1.2]

45. $f(x,f(f(x,y),f(z,f(x,z)))) \ = \ f(x,y)$ [25.1.1.2.2 ← 39.1.1]

46. $f(x,f(y,z)) \ = \ f(x,f(z,y))$ [44.1.1 ← 19.1.1]

47. $f(f(x,y),z) \ = \ f(z,f(y,x))$ [46.1.1 ← 2.1.2]

48. $f(x,f(y,z)) \ = \ f(f(z,y),x)$ [46.1.2 ← 2.1.2]

49. $f(f(x,y),f(z,u)) \ = \ f(f(u,z),f(y,x))$ [47.1.1 ← 46.1.2]

50. $f(f(x,y),f(x,f(x,f(z,y)))) \ = \ f(f(x,y),f(x,y))$ [3.1.1 → 18.1.1.2 (flip)]

51. $f(f(f(x,y),f(x,y)),f(x,x)) \ = \ f(x,f(x,x))$ [41.1.2 ← 48.1.2]

52. $f(f(f(x,x),f(x,x)),f(f(x,y),f(z,f(x,z))))$
$\quad = \ f(f(f(x,y),f(z,f(x,z))),f(z,f(x,z)))$ [30.1.1 → 42.1.1.2 (flip)]

53. $f(f(x,f(y,f(z,y))),f(x,f(x,f(z,z))))$
$\quad = \ f(f(x,f(y,f(z,y))),f(x,f(y,f(z,y))))$ [50.1.1.2.2.2 ← 29.1.1]

54. $f(f(x,f(y,z)),f(x,f(u,f(y,x)))) = x$ [27.1.2 ← 12.1.1]

55. $f(x,f(y,f(x,f(z,f(y,x))))) = f(x,x)$ [54.1.1 → 11.1.1.1 (flip)]

56. $f(x,f(y,f(z,f(x,y)))) = f(x,f(y,y))$ [55.1.1 → 18.1.1.2 (flip)]

57. $f(x,f(f(y,y),f(z,f(x,f(x,y))))) = f(x,f(f(y,y),f(y,y)))$
[56.1.1.2.2.2 ← 35.1.2]

58. $f(x,f(y,f(f(y,x),z))) = f(x,f(y,y))$ [56.1.1.2.2 ← 48.1.1]

59. $f(f(x,y),f(f(x,y),f(x,y))) = f(x,f(x,x))$ [43.1.1 ← 51.1.1 (flip)]

60. $f(f(x,f(y,y)),f(f(f(y,x),x),f(f(y,x),x)))$
$= f(f(y,x),f(f(y,x),f(y,x)))$ [59.1.1.1 ← 38.1.1]

61. $f(x,f(f(y,y),f(z,f(x,f(x,y))))) = f(x,y)$ [57.1.2.2 ← 23.1.1]

62. $f(f(x,f(y,y)),f(f(x,f(y,y)),f(f(y,x),x)))$
$= f(f(y,x),f(f(y,x),f(y,x)))$ [60.1.1.2.1 ← 38.1.1]

63. $f(f(x,f(y,y)),f(f(x,f(y,y)),f(x,f(y,y))))$
$= f(f(y,x),f(f(y,x),f(y,x)))$ [62.1.1.2.2 ← 38.1.1]

64. $f(f(x,y),f(f(x,y),f(x,y))) = f(y,f(y,y))$ [63.1.1 ← 59.1.1 (flip)]

65. $f(x,f(x,x)) = f(y,f(y,y))$ [64.1.1 ← 59.1.1]

66. $f(f(x,f(y,f(z,y))),f(x,f(y,f(z,y)))) = f(f(x,f(y,f(z,y))),f(x,z))$
[53.1.1.2 ← 37.1.1 (flip)]

67. $f(f(f(x,x),f(x,x)),f(f(x,y),f(z,f(x,z))))$
$= f(f(z,f(x,z)),f(f(x,y),f(x,y)))$ [52.1.2 ← 38.1.1]

68. $f(f(x,f(y,x)),f(f(y,z),f(y,z))) = f(y,f(f(y,z),f(x,f(y,x))))$
[67.1.1.1 ← 23.1.1 (flip)]

69. $f(f(x,f(y,x)),f(f(y,z),f(y,z))) = f(y,z)$ [68.1.2 ← 45.1.1]

* 70. $f(x,f(y,f(y,y))) = f(x,x)$ [65.1.1 → 25.1.1.2] (Sheffer 2)

71. $f(f(x,y),f(x,f(z,f(z,z)))) = x$ [65.1.1 → 6.1.1.2.2]

72. $f(f(x,f(f(y,z),x)),f(y,f(f(y,z),f(y,f(u,f(u,u))))))$
$= f(f(y,z),f(y,f(u,f(u,u))))$ [71.1.1 → 69.1.1.2.1]

73. $f(f(x,f(f(y,z),x)),f(y,y)) = f(f(y,z),f(y,f(u,f(u,u))))$
[72.1.1.2.2 ← 71.1.1]

74. $f(f(x,f(f(y,z),x)),f(y,y)) = y$ [73.1.2 ← 71.1.1]

75. $f(f(x,f(x,f(y,z))),f(z,z)) = z$ [74.1.1.1.2 ← 48.1.2]

76. $f(x,f(f(y,f(y,f(z,x))),x)) = f(y,f(y,f(z,x)))$ [75.1.1 → 8.1.1.1]

77. $f(x,f(y,f(z,f(z,f(u,f(y,x)))))) = f(x,f(y,y))$ [76.1.1 → 58.1.1.2.2]

78. $f(x,f(y,f(y,f(z,f(x,y))))) = f(x,f(y,f(x,x)))$ [77.1.1 → 18.1.1.2 (flip)]

79. $f(x,f(y,f(y,f(z,f(x,y))))) = f(x,x)$ [78.1.2 ← 26.1.1]

80. $f(f(x,x),f(y,f(y,f(z,f(y,f(x,y)))))) = f(f(x,x),f(x,x))$
[79.1.1.2.2.2.2 ← 42.1.2]

81. $f(f(x,x),f(y,f(y,f(z,f(y,f(x,y)))))) = x$ `[80.1.2 ← 24.1.1]`

82. $f(x,f(f(f(y,f(x,f(z,x))),f(y,f(x,f(z,x)))),z))$
$= f(x,f(y,f(x,f(z,x))))$ `[81.1.1 → 61.1.1.2.2]`

83. $f(x,f(f(f(y,f(x,f(z,x))),f(y,z)),z)) = f(x,f(y,f(x,f(z,x))))$
`[82.1.1.2.1 ← 66.1.1]`

84. $f(x,f(y,f(x,f(z,x)))) = f(x,f(y,z))$ `[83.1.1.2 ← 22.1.1 (flip)]`

85. $f(x,f(y,f(x,f(x,z)))) = f(x,f(y,z))$ `[84.1.1.2.2.2 ← 2.1.2]`

86. $f(x,f(y,f(x,f(z,z)))) = f(x,f(y,z))$ `[84.1.1.2.2 ← 39.1.2]`

87. $f(x,f(y,f(z,x))) = f(x,f(y,f(z,z)))$ `[85.1.1.2.2 ← 32.1.1]`

88. $f(f(x,f(y,z)),f(x,f(y,z))) = f(f(x,f(y,z)),f(x,f(z,z)))$
`[87.1.1.2 ← 19.1.1]`

89. $f(f(x,f(y,f(x,f(z,z)))),f(x,f(z,z))) = f(f(x,f(y,z)),f(x,f(y,z)))$
`[88.1.2.1 ← 86.1.2 (flip)]`

90. $f(f(x,f(y,y)),f(x,f(z,f(x,f(y,y))))) = f(f(x,f(z,y)),f(x,f(z,y)))$
`[89.1.1 ← 2.1.2]`

91. $f(f(x,f(y,y)),f(x,f(z,z))) = f(f(x,f(z,y)),f(x,f(z,y)))$
`[90.1.1 ← 87.1.1]`

* 92. $f(f(f(x,x),y),f(f(z,z),y)) = f(f(y,f(x,z)),f(y,f(x,z)))$
`[91.1.1 ← 49.1.2]` (flip of Sheffer 3)

We have proved that *many* of the pairs of equations from the original list suffice. The following two theorems, which are updated versions of Theorems 2 and 5, respectively, from the original report, are a small representative sample.

Theorem 8. The pair consisting of Commutativity and *any one* of the other equations in the list forms a basis.

Theorem 9. The pair consisting of Axiom 17 and *any one* of the other equations in the list forms a basis.

4 Further Work

We continue to accumulate proofs for new pairs of equations from the original list. Although we have been successful with many of the pairs we have tried, some pairs have eluded us. For example, we have so far been unable to prove that Axioms 7 and 8 together suffice. Whether these two equations form a 2-basis remains unknown to us.

We also continue to search for other simple axiom systems, including short single axioms in terms of the Sheffer stroke and axiom systems in terms of other sets of operations.

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