Solution to a Challenge Problem in HBCK

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Abstract

In this note, we present a first-order proof that equation

$$(((x * y) * y) * x) * x = (((y * x) * x) * y) * y$$

holds in the quasivariety **HBCK**.

1 Introduction

BCK-algebras were introduced by Iseki and Tanaka in 1976 [2], and numerous results have appeared since that time. A recent paper by Dudek [1] presents a summary of results about specific subalgebras and poses several new problems. In this note, we present a first-order proof that the equation

$$(((x * y) * y) * x) * x = (((y * x) * x) * y) * y \tag{J}$$

holds in the quasivariety **HBCK**, which can be defined with the following set of axioms.[1]

$$x * 1 = 1 \tag{M3}$$

$$1 * x = x \tag{M4}$$

$$(x * y) * ((z * x) * (z * y)) = 1 \tag{M5}$$

$$(x * y = 1 \wedge y * x = 1) \rightarrow x = y \tag{M7}$$

$$x * x = 1 \tag{M8}$$

$$x * (y * z) = y * (x * z) \tag{M9}$$

$$(x * y) * (x * z) = (y * x) * (y * z) \tag{H}$$

The problem of finding a first-order proof is posed as an open question in [4]. According to McCune and Padmanabhan ([4, p. 213]), Blok and Ferreirim indicate that such a proof necessarily exists, but the result relies on a model-theoretic argument.

[1] Axiom M4 is dependent, since it is derivable from {M3,M7,M8,M9}.

2 Proof of Equation J

Let $g(x, y)$ denote the expression $(((x * y) * y) * x) * x$. It suffices to prove that $g(x, y)$ is commutative.

In the following (machine-oriented) proof, the justification $[i \to j]$ indicates paramodulation from i into j, that is, unifying the left-hand side of i with a subterm of j, instantiating j with the corresponding substitution, and replacing the subterm with the corresponding instance of the right-hand side of i. In addition, implicit in the justifications $[i, j, k]$ of steps 25 and 55, is instantiation of i, j, and k.

1.	$x * 1 \ = \ 1$	[M3]
2.	$1 * x \ = \ x$	[M4]
3.	$(x * y) * ((z * x) * (z * y)) \ = \ 1$	[M5]
4.	$(x * y = 1 \ \wedge \ y * x = 1) \ \to \ x = y$	[M7]
5.	$x * x \ = \ 1$	[M8]
6.	$x * (y * z) \ = \ y * (x * z)$	[M9]
7.	$(x * y) * (x * z) \ = \ (y * x) * (y * z)$	[H]
8.	$(((x * y) * y) * x) * x \ = \ g(x, y)$	[Definition]
9.	$((x * y) * ((z * x) * (z * y))) * u \ = \ u$	$[3 \to 2]$
10.	$x * (y * y) \ = \ 1$	$[5 \to 1]$
11.	$x * ((x * y) * y) \ = \ 1$	$[5 \to 6]$
12.	$(x * y) * (z * ((z * x) * y)) \ = \ 1$	$[6 \to 3]$
13.	$(x * y) * (x * z) \ = \ y * ((y * x) * z)$	$[6 \to 7]$
14.	$x * (y * x) \ = \ 1$	$[6 \to 10]$
15.	$(x * (y * x)) * z \ = \ z$	$[14 \to 2]$
16.	$((x * y) * y) * g(x, y) \ = \ 1$	$[8 \to 11]$
17.	$(((x * y) * y) * z) * (1 * (x * z)) \ = \ 1$	$[11 \to 3]$
18.	$(g(x, y) * ((x * y) * y)) * (g(x, y) * z) \ = \ 1 * (((x * y) * y) * z)$	$[16 \to 7]$
19.	$x * ((y * z) * ((x * y) * z)) \ = \ 1$	$[6 \to 12]$
20.	$x * (((y * x) * z) * z) \ = \ 1$	$[15 \to 19]$
21.	$x * ((x * y) * ((y * z) * z)) \ = \ 1$	$[6 \to 19]$
22.	$x * ((x * y) * z) \ = \ y * ((y * x) * z)$	$[6 \to 13]$
23.	$(x * y) * (((z * ((z * x) * y)) * u) * u) \ = \ 1$	$[13 \to 20]$
24.	$(((x * y) * y) * z) * (x * z) \ = \ 1$	$[2 \to 17]$
25.	$((x * y) * y) * y \ = \ x * y$	[4, 11, 24]
26.	$((x * y) * y) * x \ = \ g(x, y) * x$	$[8 \to 25]$
27.	$(g(x, y) * x) * x \ = \ g(x, y)$	$[26 \to 8]$
28.	$((g(x, y) * x) * z) * g(x, y) \ = \ z * ((z * (g(x, y) * x)) * x)$	$[27 \to 13]$
29.	$x * ((y * ((y * x) * z)) * ((((x * y) * z) * u) * u)) \ = \ 1$	$[22 \to 21]$
30.	$(((x * y) * y) * z) * (((x * (1 * z)) * u) * u) \ = \ 1$	$[11 \to 23]$
31.	$((x * y) * z) * (((y * z) * u) * u) \ = \ 1$	$[15 \to 23]$
32.	$((x * y) * z) * (((u * x) * y) * z) \ = \ 1$	$[6 \to 31]$
33.	$((x * y) * y) * g(y, x) \ = \ 1$	$[8 \to 32]$
34.	$g(x, y) * ((g(x, y) * ((y * x) * x)) * z) \ = \ ((y * x) * x) * (1 * z)$	$[33 \to 22]$

35. $(((x*y)*y)*z)*(((x*z)*u)*u) \;=\; 1$ [2 → 30]
36. $((((x*y)*z)*z)*y)*g(x,y) \;=\; 1$ [8 → 35]
37. $((x*y)*z)*((z*((z*((x*y)*z))*y))*(1*g(x,y))) \;=\; 1$ [36 → 29]
38. $((x*y)*(g(x,y)*y))*(g(x,y)*z) \;=\; 1*(((x*y)*y)*z)$ [6 → 18]
39. $g(x,y)*(((y*x)*(g(x,y)*x))*z) \;=\; ((y*x)*x)*(1*z)$ [6 → 34]
40. $((x*y)*z)*((z*y)*(1*g(x,y))) \;=\; 1$ [15 → 37]
41. $((x*y)*z)*((z*y)*g(x,y)) \;=\; 1$ [2 → 40]
42. $(x*y)*(((z*y)*x)*g(z,y)) \;=\; 1$ [6 → 41]
43. $((x*y)*z)*(1*(((u*z)*y)*g(u,z))) \;=\; 1$ [42 → 31]
44. $((x*y)*z)*(((u*z)*y)*g(u,z)) \;=\; 1$ [2 → 43]
45. $((x*y)*(g(x,y)*y))*(g(x,y)*z) \;=\; ((x*y)*y)*z$ [2 → 38]
46. $((g(x,y)*x)*((x*y)*(g(x,y)*y)))*g(x,y)$
$\;=\; ((x*y)*(g(x,y)*y))*((((x*y)*y)*x)*x)$ [45 → 28]
47. $((x*y)*(g(y,x)*y))*(g(y,x)*z) \;=\; ((x*y)*y)*(1*z)$ [6 → 39]
48. $((x*y)*(g(y,x)*y))*(g(y,x)*z) \;=\; ((x*y)*y)*z$ [2 → 47]
49. $((x*y)*(g(x,y)*y))*((((x*y)*y)*x)*x)$
$\;=\; ((x*y)*((g(x,y)*x)*(g(x,y)*y)))*g(x,y)$ [6 → 46]
50. $((x*y)*(g(x,y)*y))*((((x*y)*y)*x)*x) \;=\; g(x,y)$ [9 → 49]
51. $((x*y)*(g(x,y)*y))*g(x,y) \;=\; g(x,y)$ [8 → 50]
52. $((x*(g(y,z)*z))*z)*g(y,z) \;=\; 1$ [51 → 44]
53. $((((x*y)*y)*x)*x)*g(y,x) \;=\; 1$ [48 → 52]
54. $g(x,y)*g(y,x) \;=\; 1$ [8 → 53]
55. $g(x,y) \;=\; g(y,x)$ [4, 54, 54]

This completes the proof.

Note. This proof was obtained with the assistance of the automated reasoning program Otter [3], using the method of proof sketches [5].

References

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