

## Solution to a Challenge Problem in HBCK

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### Abstract

In this note, we present a first-order proof that equation

$$(((x * y) * y) * x) * x = (((y * x) * x) * y) * y$$

holds in the quasivariety **HBCK**.

## 1 Introduction

BCK-algebras were introduced by Iseki and Tanaka in 1976 [2], and numerous results have appeared since that time. A recent paper by Dudek [1] presents a summary of results about specific subalgebras and poses several new problems. In this note, we present a first-order proof that the equation

$$(((x * y) * y) * x) * x = (((y * x) * x) * y) * y \quad (\text{J})$$

holds in the quasivariety **HBCK**, which can be defined with the following set of axioms.<sup>1</sup>

$$x * 1 = 1 \quad (\text{M3})$$

$$1 * x = x \quad (\text{M4})$$

$$(x * y) * ((z * x) * (z * y)) = 1 \quad (\text{M5})$$

$$(x * y = 1 \wedge y * x = 1) \rightarrow x = y \quad (\text{M7})$$

$$x * x = 1 \quad (\text{M8})$$

$$x * (y * z) = y * (x * z) \quad (\text{M9})$$

$$(x * y) * (x * z) = (y * x) * (y * z) \quad (\text{H})$$

The problem of finding a first-order proof is posed as an open question in [4]. According to McCune and Padmanabhan ([4, p. 213]), Blok and Ferreirim indicate that such a proof necessarily exists, but the result relies on a model-theoretic argument.

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<sup>1</sup>Axiom M4 is dependent, since it is derivable from {M3,M7,M8,M9}.

## 2 Proof of Equation J

Let  $g(x, y)$  denote the expression  $((x * y) * y) * x$ . It suffices to prove that  $g(x, y)$  is commutative.

In the following (machine-oriented) proof, the justification  $[i \rightarrow j]$  indicates paramodulation from  $i$  into  $j$ , that is, unifying the left-hand side of  $i$  with a subterm of  $j$ , instantiating  $j$  with the corresponding substitution, and replacing the subterm with the corresponding instance of the right-hand side of  $i$ . In addition, implicit in the justifications  $[i, j, k]$  of steps 25 and 55, is instantiation of  $i, j$ , and  $k$ .

1.  $x * 1 = 1$  [M3]
2.  $1 * x = x$  [M4]
3.  $(x * y) * ((z * x) * (z * y)) = 1$  [M5]
4.  $(x * y = 1 \wedge y * x = 1) \rightarrow x = y$  [M7]
5.  $x * x = 1$  [M8]
6.  $x * (y * z) = y * (x * z)$  [M9]
7.  $(x * y) * (x * z) = (y * x) * (y * z)$  [H]
8.  $((x * y) * y) * x = g(x, y)$  [Definition]
9.  $((x * y) * ((z * x) * (z * y))) * u = u$  [3  $\rightarrow$  2]
10.  $x * (y * y) = 1$  [5  $\rightarrow$  1]
11.  $x * ((x * y) * y) = 1$  [5  $\rightarrow$  6]
12.  $(x * y) * (z * ((z * x) * y)) = 1$  [6  $\rightarrow$  3]
13.  $(x * y) * (x * z) = y * ((y * x) * z)$  [6  $\rightarrow$  7]
14.  $x * (y * x) = 1$  [6  $\rightarrow$  10]
15.  $(x * (y * x)) * z = z$  [14  $\rightarrow$  2]
16.  $((x * y) * y) * g(x, y) = 1$  [8  $\rightarrow$  11]
17.  $((x * y) * y) * z * (1 * (x * z)) = 1$  [11  $\rightarrow$  3]
18.  $(g(x, y) * ((x * y) * y)) * (g(x, y) * z) = 1 * (((x * y) * y) * z)$  [16  $\rightarrow$  7]
19.  $x * ((y * z) * ((x * y) * z)) = 1$  [6  $\rightarrow$  12]
20.  $x * ((y * x) * z) * z = 1$  [15  $\rightarrow$  19]
21.  $x * ((x * y) * ((y * z) * z)) = 1$  [6  $\rightarrow$  19]
22.  $x * ((x * y) * z) = y * ((y * x) * z)$  [6  $\rightarrow$  13]
23.  $(x * y) * ((z * ((z * x) * y)) * u) * u = 1$  [13  $\rightarrow$  20]
24.  $((x * y) * y) * z * (x * z) = 1$  [2  $\rightarrow$  17]
25.  $((x * y) * y) * y = x * y$  [4, 11, 24]
26.  $((x * y) * y) * x = g(x, y) * x$  [8  $\rightarrow$  25]
27.  $(g(x, y) * x) * x = g(x, y)$  [26  $\rightarrow$  8]
28.  $((g(x, y) * x) * z) * g(x, y) = z * ((z * (g(x, y) * x)) * x)$  [27  $\rightarrow$  13]
29.  $x * ((y * ((y * x) * z)) * (((x * y) * z) * u) * u) = 1$  [22  $\rightarrow$  21]
30.  $((x * y) * y) * z * (((x * (1 * z)) * u) * u) = 1$  [11  $\rightarrow$  23]
31.  $((x * y) * z) * (((y * z) * u) * u) = 1$  [15  $\rightarrow$  23]
32.  $((x * y) * z) * (((u * x) * y) * z) = 1$  [6  $\rightarrow$  31]
33.  $((x * y) * y) * g(y, x) = 1$  [8  $\rightarrow$  32]
34.  $g(x, y) * ((g(x, y) * ((y * x) * x)) * z) = ((y * x) * x) * (1 * z)$  [33  $\rightarrow$  22]

35.  $((x * y) * y) * z * ((x * z) * u) * u = 1$  [2 → 30]
36.  $((x * y) * z) * z * y * g(x, y) = 1$  [8 → 35]
37.  $(x * y) * z * (z * ((z * ((x * y) * z)) * y)) * (1 * g(x, y)) = 1$  [36 → 29]
38.  $(x * y) * (g(x, y) * y) * (g(x, y) * z) = 1 * (((x * y) * y) * z)$  [6 → 18]
39.  $g(x, y) * (((y * x) * (g(x, y) * x)) * z) = ((y * x) * x) * (1 * z)$  [6 → 34]
40.  $(x * y) * z * ((z * y) * (1 * g(x, y))) = 1$  [15 → 37]
41.  $(x * y) * z * ((z * y) * g(x, y)) = 1$  [2 → 40]
42.  $(x * y) * ((z * y) * x) * g(z, y) = 1$  [6 → 41]
43.  $(x * y) * z * (1 * (((u * z) * y) * g(u, z))) = 1$  [42 → 31]
44.  $(x * y) * z * (((u * z) * y) * g(u, z)) = 1$  [2 → 43]
45.  $(x * y) * (g(x, y) * y) * (g(x, y) * z) = ((x * y) * y) * z$  [2 → 38]
46.  $((g(x, y) * x) * ((x * y) * (g(x, y) * y))) * g(x, y)$   
 $= ((x * y) * (g(x, y) * y)) * (((x * y) * y) * x) * x$  [45 → 28]
47.  $(x * y) * (g(y, x) * y) * (g(y, x) * z) = ((x * y) * y) * (1 * z)$  [6 → 39]
48.  $(x * y) * (g(y, x) * y) * (g(y, x) * z) = ((x * y) * y) * z$  [2 → 47]
49.  $(x * y) * (g(x, y) * y) * (((x * y) * y) * x) * x$   
 $= ((x * y) * ((g(x, y) * x) * (g(x, y) * y))) * g(x, y)$  [6 → 46]
50.  $(x * y) * (g(x, y) * y) * (((x * y) * y) * x) * x = g(x, y)$  [9 → 49]
51.  $(x * y) * (g(x, y) * y) * g(x, y) = g(x, y)$  [8 → 50]
52.  $(x * (g(y, z) * z)) * z * g(y, z) = 1$  [51 → 44]
53.  $((x * y) * y) * x * g(y, x) = 1$  [48 → 52]
54.  $g(x, y) * g(y, x) = 1$  [8 → 53]
55.  $g(x, y) = g(y, x)$  [4, 54, 54]

This completes the proof.

*Note.* This proof was obtained with the assistance of the automated reasoning program Otter [3], using the method of proof sketches [5].

## References

- [1] W. Dudek. Unsolved problems in BCK-algebras. *East Asian Math J.*, 17:115–128, 2001.
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