# An Exact Characterization of Greedy Structures

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**ABSTRACT**

We present exact characterizations of structures on which the greedy algorithm produces optimal solutions. Our characterization, which we call matroid embeddings, complete the partial characterizations of Rado, Gale, and Edmonds (matroids), and of Korte and Lovasz (greedoids). We show that the greedy algorithm optimizes all linear objective functions if and only if the problem structure (phrased in terms of either accessible set systems or hereditary languages) is a matroid embedding. We also present an exact characterization of the objective functions optimized by the greedy algorithm on matroid embeddings. Finally, we present an exact characterization of the structures on which the greedy algorithm optimizes all bottleneck functions, structures which are less constrained than matroid embeddings.

## 1 Introduction

Obtaining an exact characterization of the class of problems for which the greedy algorithm returns an optimal solution has been an open problem. Rado [9], Gale [3], and Edmonds [1] independently showed that matroids characterize a subclass of problems on which the greedy algorithm always optimizes linear objectives; their results are limited by the assumption that the greedy algorithm operates on a hereditary set system, whereas most common greedy algorithms operate on set systems that do not obey the heredity axiom. Faigle [2] has provided an exact characterization of the partially ordered set systems on which the greedy algorithm optimizes linear objectives, but the assumption of a partial order, which constrains the choices of the greedy algorithm, limits the characterization. Korte and Lovasz [6, 7] have defined greedoids, a generalization of matroids, and have provided necessary and sufficient conditions for the greedy algorithm to be optimal with respect to linear objectives when run on greedoids. But greedoids are both too general (the greedy algorithm need not return an optimal solution on a greedoid) and too constraining: there exist set systems on which the greedy algorithm always optimizes linear objectives, but which are not greedoids. Goecke [4] has given necessary and sufficient conditions for the optimization of linear objectives over set systems by a variant of the greedy algorithm, but his variant of the greedy algorithm (find any solution, partial or complete, which optimizes the objective) does not fit well in many standard applications of the greedy algorithm, in particular, applications where the objective function is to be minimized.

We solve the open problem by presenting three exact characterizations, all based on a very general model of the problem structure:

1. an exact characterization, which we call a *matroid embedding*, of the structure of problems on which the greedy algorithm optimizes all linear objectives;
2. a similar characterization for bottleneck objectives; and
3. an exact characterization of the objective functions optimized by the greedy algorithm on matroid embeddings.

Our presentation is in four parts. First we set the stage by briefly recalling the definitions and main existing results pertaining to set systems and the greedy algorithm. Next we introduce additional properties, relate them to the existing structures, and prove our main result. In a third part, we extend these results to a family of objective functions and then examine one particular class, the bottleneck functions, and give an exact characterization of the problems on which the greedy algorithm optimizes these objectives. The fourth part extends our results from set systems to languages. We conclude with some general observations and a number of open questions.

## 2 Preliminaries

We include this section for readers unfamiliar with the terminology; other readers may wish to skip to the next section.

Let $S$ be a set and $\mathcal{C}$ a collection of subsets of $S$; the pair $(S, \mathcal{C})$ is called a *set system*. In order to simplify the notation, we let $ext(X) = \{\, x \mid X \cup \{x\} \in \mathcal{C} \,\}$. A set system is an *accessible set system* if it obeys the two axioms:

*(trivial axiom)* $\emptyset \in \mathcal{C}$

*(accessibility axiom)* If $X \in \mathcal{C}$ and $X \neq \emptyset$, then $\exists x \in X$ such that $X - \{x\} \in \mathcal{C}$.

In an accessible set system, $(S, \mathcal{C})$, the elements of $\mathcal{C}$ are called *feasible sets*; a maximal feasible set (i.e., one which is not contained in any other) is called a *basis*. A set system is a *hereditary set system* (also known as a *simplicial complex* or an *independence structure*) if it obeys the trivial axiom and:

*(heredity axiom)* If $X \in \mathcal{C}$ and $Y \subseteq X$, then $Y \in \mathcal{C}$.

Given an arbitrary, but non-empty set system $(S, \mathcal{C})$, we define its *hereditary closure* as the set system $(S, \mathcal{C}^*)$, where $\mathcal{C}^* = \{Y \subseteq X \mid X \in \mathcal{C}\}$.

Let $(S, \mathcal{C})$ be a set system. An *objective function* is an assignment of values to the subsets of $S$, $f\colon 2^S \to$ R. We define the *optimization problem* for $f$ over $(S, \mathcal{C})$ as the problem of finding a basis, $B \in \mathcal{C}$, such that $f(B) = \max\{\, f(X) \mid X$ is a basis of $(S, \mathcal{C})\,\}$. (Note that only bases are candidates for solution; further note that we restrict our discussion to maximization problems: *mutatis mutandis*, identical results hold for minimization problems.) Given a weight assignment to the elements of $S$, $w\colon S \to$ R, the induced *linear objective function* is defined by $f(X) = \sum_{x \in X} w(x)$, for $X \subseteq S$, and the induced *bottleneck objective function* is defined by $f(X) = \min_{x \in X} w(x)$, for $X \subseteq S$.

Informally, the greedy algorithm, when run on a set system, builds a solution by beginning with the empty set and successively adding the best remaining element while maintaining feasibility. (Korte and Lovasz [7] have considered a variant known as the *worst-out greedy algorithm*; we do not pursue it further here.) The accessibility of a set system allows any feasible set, and in particular any basis, to be built one element at a time from the empty set—a necessary condition for the greedy algorithm to succeed. Formally, we define the *best-in greedy algorithm* on an accessible set system $(S, \mathcal{C})$, with objective function $f\colon 2^S \to$ R, as follows. The algorithm starts with the empty set, $\emptyset$; at each step, $i$, it chooses an element $x_i \in S$ such that

1. $\{x_1, x_2, \ldots, x_i\} \in \mathcal{C}$; and
2. $f\big(\{x_1, \ldots, x_i\}\big) = \max\big\{f\big(\{x_1, \ldots, x_{i-1}, y\}\big) \bigm| \{x_1, \ldots, x_{i-1}, y\} \in \mathcal{C}\big\}$;

the algorithm terminates when it can no longer incorporate another element into its partial solution, i.e., when $ext\{x_1, \ldots, x_{i-1}\} = \emptyset$.

**Definition 1.** A feasible set $X$ is a *greedy set* under $f$ if there exists a sequence $\emptyset, \{x_1\}, \{x_1, x_2\}, \ldots, \{x_1, \ldots, x_i\}, \ldots, \{x_1, \ldots, x_i, \ldots, x_k\} = X$ of feasible subsets of $X$ such that, for each $i$, $f(\{x_1, \ldots, x_{i-1}, x_i\}) = \max\{f(\{x_1, \ldots, x_{i-1}, y\}) \mid \{x_1, \ldots, x_{i-1}, y\}$ is feasible$\}$. A basis with this property is called a *greedy basis*. □

The greedy algorithm, for any objective function $f$, can construct only greedy sets under $f$; using the proper tie-breaking rule, it can construct any greedy set under $f$. We say that an accessible set system $(S, \mathcal{C})$ is *pathological* if there exist feasible sets $A$ and $B$, with $A \subset B$, such that $B$ is a basis and $ext(A) = \emptyset$. Due to the presence of pathologies, the greedy algorithm can terminate at a set that is not a basis. One could, as a result, redefine the optimization problem as the problem of finding a non-extensible set of maximal value. Our results hold under this interpretation as well.

A *greedoid* is an accessible set system $(S, \mathcal{C})$ that obeys the following axiom:

*(augmentation axiom)* If $X, Y \in \mathcal{C}$ and $|X| = |Y| + 1$, then $\exists x \in X - Y$ such that $Y \cup \{x\} \in \mathcal{C}$.

A *matroid* is a hereditary set system that obeys the augmentation axiom. (Note that the bases of a greedoid or matroid have equal cardinality and that pathologies cannot occur.) This axiom is often phrased more generally:

*(exchange axiom)* If $X, Y \in \mathcal{C}$ and $|Y| < |X|$, then $\exists x \in X - Y$ such that $Y \cup \{x\} \in \mathcal{C}$.

In the presence of the trivial axiom, the exchange axiom is equivalent to the combination of the accessibility and augmentation axioms.

Rado [9], Gale [3], and Edmonds [1] independently proved that the best-in greedy algorithm optimizes all linear objective functions over a hereditary set system $(S, \mathcal{C})$ if and only if $(S, \mathcal{C})$ is a matroid. Korte and Lovasz [6, 7] defined greedoids and proved that the best-in greedy algorithm optimizes all linear objective functions over a greedoid $(S, \mathcal{C})$ if and only if $(S, \mathcal{C})$ obeys the following axiom:

*(strong exchange axiom)* Let $A, B \in \mathcal{C}$, with $B$ a basis and $A \subset B$. If $x \in S - B$ is such that $A \cup \{x\} \in \mathcal{C}$, then $\exists y \in B - A$ such that $A \cup \{y\} \in \mathcal{C}$ and $B \cup \{x\} - \{y\} \in \mathcal{C}$.

# 3 An Exact Characterization

We propose two new axioms in order to establish an exact characterization. The first is a strengthened version of accessibility for bases. An accessible set system, $(S, \mathcal{C})$, is *extensible* if it obeys the following axiom:

*(extensibility axiom)* If $X$ and $B$ are feasible sets, with $B$ a basis and $X \subset B$, then there exists $y \in B - X$ such that $X \cup \{y\}$ is feasible.

Note that every greedoid is extensible. An accessible set system, $(S, \mathcal{C})$, is *closure-congruent* if it obeys the following axiom:

*(closure-congruence axiom)* $\forall X \in \mathcal{C}, \forall x, y \in ext(X), \forall E \subseteq S - X - ext(X), X \cup \{x\} \cup E \in \mathcal{C}^* \implies X \cup \{y\} \cup E \in \mathcal{C}^*$.

**Example 1.** Consider an accessible set system that models the minimum spanning tree problem on some connected graph $G$. The ground set is the set of edges of $G$ and the feasible sets are the subtrees of $G$. The bases of $(S, \mathcal{C})$ are then the spanning trees of $G$. If $T$ is a feasible set, then we have $ext(T) = \{ e \mid e$ has exactly one endpoint in $T\}$. This set system is a generalization of the setting of Prim's algorithm (in which feasible sets are all subtrees of $G$ that include a designated vertex), yet remains more restrictive than the setting of Kruskal's algorithm (in which the feasible sets are the subforests of $G$). For this set system, the closure-congruence axiom reduces to the assertion:

> Let $T$ be any subtree of $G$, $e$ and $e'$ two edges in $ext(T)$, and $E \subseteq S - T - ext(T)$ any collection of edges; then $T \cup \{e\} \cup E$ is a subforest of $G$ (and hence in the hereditary closure of $(S, \mathcal{C})$) if and only if $T \cup \{e'\} \cup E$ is a subforest of $G$.

It is readily verified that this condition is obeyed. For any $E \subseteq S - T - ext(T)$, each of $T \cup \{e\} \cup E$ and $T \cup \{e'\} \cup E$ is a subforest if and only if $E$ is an acyclic collection of edges, each member of which is an edge with neither endpoint in $T$. □

Although every hereditary set system is closure-congruent (because, in a hereditary set system, the empty set is the only choice for $E$ in the definition of closure-congruence), not every greedoid is closure-congruent.

**Example 2.** Let $(S, \mathcal{C})$ be given by $S = \{a, b, c\}$ and $\mathcal{C} = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$; this set system is a greedoid, but it is not closure-congruent. The hereditary closure of this greedoid is the matroid given by $\mathcal{C} = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}\}$. The greedoid and its closure are shown below.


*the greedoid* *its hereditary closure*

To see that the closure-congruence axiom fails on this set system, consider $X = \emptyset$. Now, $ext(\emptyset) = \{a, b\}$, and take $E = \{c\} \subseteq S - \emptyset - \{a, b\}$. Since $\emptyset \cup \{b\} \cup E = \{b, c\}$ is in $\mathcal{C}$ and *a fortiori* in the closure, $\emptyset \cup \{a\} \cup E = \{a, c\}$ should be in the closure, but it is not. □

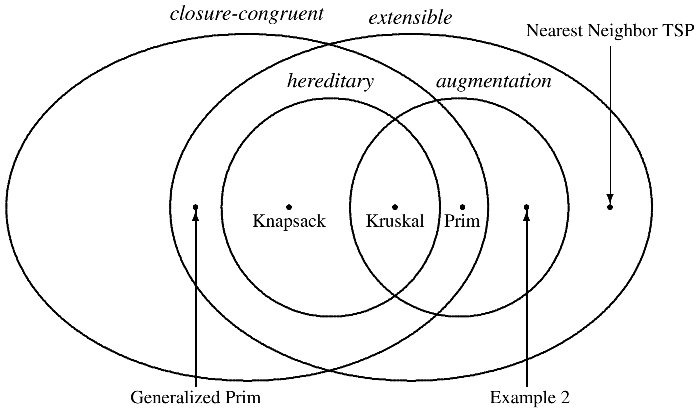



**Figure 1**: The Relationships Among Varieties of Accessible Set Systems


**Figure 2**: The Weight Assignments Used in Proving that $\mathcal{C}^*$ is a Matroid

Greedoids that obey the strong exchange axiom are closure-congruent (a corollary of Theorem 1), but there exist extensible, closure-congruent accessible set systems that do not obey the augmentation axiom and hence do not define greedoids: the set system of Example 1 ("Generalized Prim") is one such.

**Definition 2.** A *matroid embedding* is an accessible set system which is extensible, closure-congruent, and the hereditary closure of which is a matroid. □

Our previous observations about Example 1 show that it is a matroid embedding, yet not a greedoid. Indeed, the three conditions defining a matroid embedding are independent. Figure 1 shows the relationships among our axioms and previously defined structures.

We can now prove our main result, which solves the open problem.

**Theorem 1.** Let $(S, \mathcal{C})$ be an accessible set system; then the following are equivalent:

1. For every positive weighted linear objective function, $(S, \mathcal{C})$ has an optimal greedy basis.
2. $(S, \mathcal{C})$ is a matroid embedding.
3. For every linear objective function, the greedy bases of $(S, \mathcal{C})$ are exactly its optimal bases.

□

**Proof:** We prove the implications $(1) \Longrightarrow (2)$ and $(2) \Longrightarrow (3)$; the implication $(3) \Longrightarrow (1)$ is trivial.

$(1) \Longrightarrow (2)$ We begin by showing that $\mathcal{C}^*$ must be a matroid, a result first derived by Helman [5]. Assume two sets, $X, Y \in \mathcal{C}^*$, with $|X| = |Y| + 1$, between which augmentation fails in $\mathcal{C}^*$. Since augmentation fails, no basis that contains all of $Y$ can contain any element of $X - Y$. We design a pair of weight assignments, $w_1$ and $w_2$, such that: (i) the relative ordering of elements by weight is the same under both $w_1$ and $w_2$ and distinct elements get assigned distinct weights in each weight assignment—so that $w_1$ and $w_2$ share the same unique non-extensible greedy set; and (ii) $w_1$ and $w_2$ share no optimal basis, thereby contradicting (1) and proving the result. Figure 2 illustrates how the two weight assignments are chosen. Observe that, under $w_1$, an optimal basis cannot contain all of $Y$, while, under $w_2$, an optimal basis must contain all of $Y$.

We now prove that $(S, \mathcal{C})$ is extensible. Let $A \subset B$, $A, B \in \mathcal{C}$, with $B$ a basis. Since $(S, \mathcal{C})$ is accessible, there exists a sequence of feasible sets $\emptyset, \{x_1\}, \{x_1, x_2\}, \ldots, \{x_1, x_2, \ldots, x_k\} = A$; denote the other elements of

$S$ by $x_{k+1}, x_{k+2}, \ldots, x_n$. We force the greedy algorithm to construct each set in the sequence leading to $A$ by assigning weights as follows:

$$w(x_i) = \begin{cases} 1 + \epsilon/i & \text{for } 1 \leq i \leq k \\ 1 & \text{for } x_i \in B - A \\ \epsilon & \text{for } x_i \in S - B. \end{cases}$$

Thus $B$ is the unique optimal basis; since the greedy algorithm must start by constructing $A$, it can construct $B$ only if $A$ is extensible to $B$, as desired.

Finally, we show that $(S, \mathcal{C})$ is closure-congruent. Let $A \in C$, $x, y \in ext(A)$, and $E \subseteq S - A - ext(A)$, with $A \cup \{x\} \cup E \in \mathcal{C}^*$. In the style of the previous construction, we force the greedy algorithm to construct $A$, followed by the set $A \cup \{y\}$, by using a weight assignment that gives very high weights to elements of $A$ and $E$, high weights to $x$ and $y$, and very low weights to all other elements. Since $A \cup \{x\} \cup E \in \mathcal{C}^*$ there is a basis $B$ containing $A \cup \{x\} \cup E$. The greedy algorithm must begin by constructing $A \cup \{y\}$ and constructs some optimal basis $B'$; but then we must have $E \subseteq B'$ and thus $A \cup \{y\} \cup E \in \mathcal{C}^*$.

$(2) \implies (3)$ (Note that the structure of a matroid embedding ensures that all non-extensible sets are bases.) Assume that, for some linear objective function $f$, some greedy basis, $B_g$, is not optimal. Let $A$ be a greedy subset of $B_g$ of maximal size, with the property that $A$ is contained in some optimal basis $B$. Since $A$ itself cannot be a basis, $ext(A)$ is not empty. Let $x$ be an element that the greedy algorithm can add to $A$. We have $A \cup \{x\} \not\subseteq B$, or else $A \cup \{x\}$ would be a greedy subset of $B_g$ contained in $B$, contradicting the maximality of $A$. By extensibility, there exists $y \in B \cap ext(A)$; set $E = B - A - ext(A)$. Observe that $A \cup \{y\} \cup E$ is in $\mathcal{C}^*$, so that, by closure-congruence, so is $A \cup \{x\} \cup E$. Because $(S, \mathcal{C}^*)$ is a matroid, we can apply the exchange axiom to $A \cup \{x\} \cup E$ with respect to $B$, yielding some basis $B'$. $B$ and $B'$ differ by one element: $B'$ contains $x$ at the expense of some other element in $ext(A)$, say $z$. Since the greedy algorithm chose to augment $A$ with $x$, we know that $w(x) \geq w(z)$, so that $f(B') \geq f(B)$ and thus $B'$ is an optimal basis. But then $A \cup \{x\} \subset B'$ is a greedy subset of $B_g$, which contradicts the maximality of $A$. A similar argument also shows that every optimal basis is greedy: if some non-greedy optimal basis, $B$, exists, let $A$ be its largest greedy subset; note that we must have $|B| \geq |A| + 2$, since otherwise $A$ can be extended to $B$ and that extension must be greedy because $B$ is optimal. But then $B'$, produced as above, has a larger objective value than $B$, because, since $A \cup \{z\} \subset B$ is not greedy, we must have $w(x) > w(z)$; hence $B$ is not optimal, yielding the desired contradiction.

*Q.E.D.*

This theorem subsumes the results of Rado [9], Gale [3], and Edmonds [1], as well as Theorem 4.2 of Korte and Lovasz [7]; more importantly, unlike these results, it provides an exact structural characterization of the problems on which the best-in greedy algorithm works for all linear objectives.

# 4 Other Classes of Objective Functions

## 4.1 Consistent Functions

We identify the largest class of functions to which the results of the previous section apply.

**Definition 3.** Let $f(\,)$ be an objective function and $S$ the ground set.

- $f(\,)$ is *consistent* if, given sets $T \subset T' \subset S$ and elements $x,\ y \in S - T'$, we have

$$f(T \cup \{x\}) \geq f(T \cup \{y\}) \implies f(T' \cup \{x\}) \geq f(T' \cup \{y\});$$

- $f(\,)$ is *strictly consistent* if we can further assert that

$$f(T \cup \{x\}) > f(T \cup \{y\}) \implies f(T' \cup \{x\}) > f(T' \cup \{y\});$$

- $f(\,)$ is *weakly consistent* if we strengthen the hypothesis used in consistency to exclude equality, i.e., if we have

$$f(T \cup \{x\}) > f(T \cup \{y\}) \implies f(T' \cup \{x\}) \geq f(T' \cup \{y\}).$$ □

Note that linear objectives are strictly consistent, while bottleneck objectives are consistent.

**Theorem 2.** Let $S$ be a set and $f(\ )$ a function defined on $2^S$.

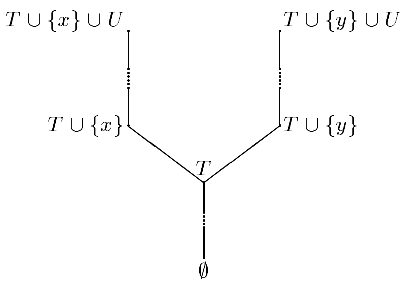
1. $f(\ )$ is strictly consistent if and only if, for each matroid embedding on $S$, the greedy bases are exactly the optimal bases.
2. $f(\ )$ is consistent if and only if, for each matroid embedding on $S$, all greedy bases are optimal.
3. $f(\ )$ is weakly consistent if and only if, for each matroid embedding on $S$, some greedy basis is optimal.

$\square$

**Proof:**

1. To show the only if part, it is enough to observe that, in the proof of Theorem 1, the inequalities involving $w(x)$ and $w(z)$ can be replaced by inequalities involving $f(A \cup \{x\})$ and $f(A \cup \{z\})$. Letting $T' = B \cap B'$, the strict consistency of $f$ leads to the same contradictions as in the previous proof.

   In order to prove the if part, assume that $f(\ )$ fails consistency (strict or not) on sets $T, U$ and elements $x, y \notin T$, with $U \cap (T \cup \{x, y\}) = \emptyset$. Consider the matroid embedding depicted below.

   If $f(\ )$ is not consistent, then we have $f(T \cup \{x\}) \geq f(T \cup \{y\})$ and yet also $f(T \cup \{x\} \cup U) < f(T \cup \{y\} \cup U)$. But then $T \cup \{x\} \cup U$ is a suboptimal greedy basis, the desired contradiction. If $f(\ )$ is not strictly consistent, then we have $f(T \cup \{x\}) > f(T \cup \{y\})$ and yet also $f(T \cup \{x\} \cup U) \leq f(T \cup \{y\} \cup U)$. But then $T \cup \{y\} \cup U$ is an optimal basis and yet is not greedy, the desired contradiction.

2. and 3. The same proof techniques apply, with the obvious changes.

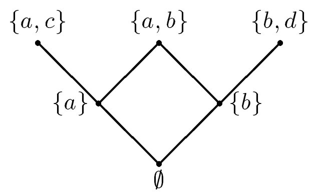
*Q.E.D.*

## 4.2 An Exact Characterization of Greedy Structures for Bottleneck Functions

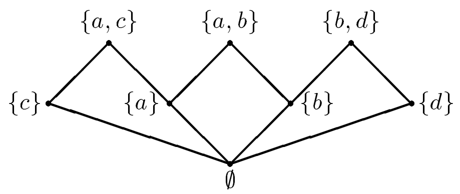
Bottlenecks functions form an important subclass of consistent functions. Formally, we define a (simple) *bottleneck function* to be an objective function of the form $f(A) = \min_{x \in A} w(x)$; by convention, we set $f(\emptyset) = 1 + \max_{x \in S} w(x)$. Our previous results show that the greedy algorithm is optimal for all bottleneck objective functions when run on a matroid embedding. Korte and Lovasz [6] considered a generalization of bottleneck objectives in which the weight of an element is a non-decreasing function of the size of the feasible set in which it could be included; they showed that the greedy algorithm is optimal for all such generalized bottleneck objectives only if the set system defines a greedoid. Since not every matroid embedding is a greedoid, this result does not hold when restricted to simple bottleneck objectives.

The matroid embedding structure is not necessary to ensure optimality of the greedy algorithm for accessible set systems with bottleneck objectives. In particular, it is easily verified that optimality for all bottleneck objectives on an accessible set system does not imply that the hereditary closure of the set system is a matroid.

**Example 3.** The simple greedoid given by $S = \{a, b, c, d\}$ with feasible sets $\emptyset$, $\{a\}$, $\{b\}$, $\{a, b\}$, $\{a, c\}$, and $\{b, d\}$, and pictured below with its hereditary closure, provides the desired counterexample.




*the greedoid*




*its hereditary closure*

The closure fails augmentation on, say, $\{c\}$ and $\{b, d\}$ and thus is not a matroid, but it is easily verified that the best-in greedy algorithm optimizes all bottleneck objectives on this set system. $\square$

We introduce a more restricted property. An accessible set system $(S, \mathcal{C})$ is *strongly extensible* if it obeys the following axiom:

*(strong extensibility axiom)* For any $X, B \in \mathcal{C}$, with $B$ a basis and $|X| < |B|$, there exists $x \in B - X$ such that $X \cup \{x\} \in \mathcal{C}$.

The bases of any strongly extensible accessible set system are of the same cardinality; in fact, a hereditary set system is strongly extensible if and only if it is a matroid.

**Lemma 1.** Let $(S, \mathcal{C})$ be an accessible set system. If, for every positive weighted bottleneck function $f$, there exists a greedy basis that is also optimal, then $(S, \mathcal{C})$ is extensible and all of its bases have equal cardinality. $\square$

**Proof:** Assume that there exists a basis $B$ and a feasible set, $A \subset B$ such that either: (i) $A$ is non-extensible; or (ii) for all $x \in ext(A)$, $A \cup \{x\} \nsubseteq B$. We force the greedy algorithm to construct $A$ by assigning suitable weights, also giving elements of $S - B$ very low weights. But now the greedy algorithm must terminate with a suboptimal feasible set. If $A$ is non-extensible, then it is the unique non-extensible greedy set, so that there does not exist an optimal greedy basis, contrary to the hypothesis of the lemma. Otherwise, because $A$ cannot be extended with an element of $B$, any greedy basis contains an element of $S - B$ and thus is not optimal.

Assume that at least two sizes of bases exist; let $C$ be an arbitrary non-minimal size basis and let $B$ be a minimal size basis, such that among all minimal size bases, $B$ shares with $C$ a largest size feasible subset, $A$. Note that $A$ is a proper subset of $B$ and cannot itself be a basis. Since $(S, \mathcal{C})$ is extensible and since $A \subset C$, there exists some $y \in ext(A)$ such that $A \cup \{y\} \subset C$; note that, by our assumption of maximality of $A$, $y \notin B$. We force the greedy algorithm to construct the set $A \cup \{y\}$ and then to complete it in suboptimal manner by assigning suitable weights, including very low weights for elements of $S - B$. Under such an assignment, basis $B$ is optimal. The greedy bases constructed must all contain $A \cup \{y\}$ and, by our assumption of the maximality of $A$, have size greater than $|B|$. Thus at least $|B - A|$ elements must be added to $A \cup \{y\}$; since $B$ is a basis, these elements cannot all come from $B$. Thus all greedy bases are suboptimal, the desired contradiction.

*Q.E.D.*

**Theorem 3.** Let $(S, \mathcal{C})$ be an accessible set system; then the following are equivalent:

1. For every positive weighted bottleneck function, $(S, \mathcal{C})$ has an optimal greedy basis.
2. $(S, \mathcal{C})$ is strongly extensible.
3. For every bottleneck function, $(S, \mathcal{C})$ has at least one greedy basis and all its greedy bases are optimal.

$\square$

**Proof:** As before, only two of the three implications are non-trivial.

$(1) \implies (2)$ Assume that there exist $A, B \in \mathcal{C}$, where $B$ is a basis, with $|A| < |B|$ and $ext(A) \cap B = \emptyset$. By the previous lemma $A$ must be extensible. We force the greedy algorithm to construct $A$ and to extend it to a suboptimal basis by assigning suitable weights, including very low weights to elements of $S - B - A$. Since $A$ cannot be extended by an element of $B$, every greedy basis has very low value, while $B$ is an optimal basis with higher value, the desired contradiction.

(2) $\implies$ (3) (Note that any non-extensible set must be a basis, by strong extensibility.) For some assignment of weights, assume that some greedy basis $B$ is suboptimal. Let $A$ be a greedy subset of $B$ of maximal size that: (i) the greedy algorithm can extend it to the greedy basis $B$; and (ii) is a subset of some optimal basis, $B'$. Let $A \cup \{x\}$ be a greedy set extensible to $B$. Since the set system is strongly extensible, there also exists $y \in B' - A$ such that $A \cup \{y\}$ is feasible, but note that $A \cup \{y\}$ cannot be a greedy set extensible to $B$, as this would contradict the maximality of $A$. Since $A \cup \{y\} \subseteq B'$, we have $f(A \cup \{y\}) \geq f(B')$; since the set system is strongly extensible (and thus allows $A \cup \{x\}$ to be extended with elements from some optimal basis), since $A$ is maximal, and since the objective function is determined by the minimum weight of its arguments, we also have $f(B') > f(A \cup \{x\})$. Combining these two inequalities yields $f(A \cup \{y\}) > f(A \cup \{x\})$, which contradicts the fact that the greedy algorithm can choose $x$.

*Q.E.D.*

# 5 Exact Characterizations of Greedy Languages

While set systems have been the traditional setting for defining and studying greedy algorithms, several researchers have recognized the desirability of extending the results to more general settings (Helman [5], Korte and Lovasz [6]). In this section, we demonstrate that our exact characterizations extend directly to hereditary languages.

In the language world, feasible structures are ordered sets, or strings, generally called *simple words*. Let $S$ be a set and $\mathcal{L}$ a collection of simple words on $S$; further let $s(\alpha)$ for each $\alpha \in \mathcal{L}$ denote the (unordered) subset of $S$ corresponding to $\alpha$ and let $s(\mathcal{L})$ denote the collection of unordered subsets corresponding to the words of $\mathcal{L}$—i.e., $s(\mathcal{L}) = \{A \mid A = s(\alpha),\ \alpha \in \mathcal{L}\}$. We say that $(S, \mathcal{L})$ is a *hereditary word system* (or *hereditary language*) if it obeys the two axioms:

*(trivial axiom)* $\mathcal{L} \neq \emptyset$

*(heredity axiom)* If $\alpha \in \mathcal{L}$ and $\beta$ is a prefix of $\alpha$ (i.e., $\alpha = \beta\gamma$ for some string $\gamma$), then also $\beta \in \mathcal{L}$.

If $(S, \mathcal{L})$ is a hereditary language, we call the elements of $\mathcal{L}$ *feasible words* and any feasible word $\alpha$ with the property that there does not exist $x \in S$ with $\alpha x \in \mathcal{L}$ is called a *basic word*. For any word $\alpha$, let $ext(\alpha)$ denote the set $\{x \mid \alpha x \in \mathcal{L}\}$. Note that, if $(S, \mathcal{L})$ is a hereditary language, then $(S, s(\mathcal{L}))$ is an accessible set system. We define the hereditary closure of a hereditary language, $(S, \mathcal{L})$, to be the hereditary closure, $(S, s^*(\mathcal{L}))$, of the corresponding accessible set system, $(S, s(\mathcal{L}))$.

There is a very natural link between hereditary languages and the greedy algorithm, as hereditary languages record the full history of the execution of the algorithm. Formally, the best-in greedy algorithm on a hereditary language $(S, \mathcal{L})$ with objective function $f\colon \mathcal{L} \to \mathrm{R}$ starts with the empty string $\lambda$; at each step $i$, it chooses an element $x_i \in S$ such that

1. $x_1 x_2 \ldots x_i \in \mathcal{L}$; and
2. $f\big(x_1 x_2 \ldots x_i\big) = \max\big\{f\big(x_1 \ldots x_{i-1} y\big) \bigm| x_1 \ldots x_{i-1} y \in \mathcal{L}\big\}$;

the algorithm terminates when it has constructed a basic word. A feasible word $x_1 x_2 \ldots x_k$ is a *greedy word* under $f$ if, for each $1 \leq i \leq k$, $f(x_1 \ldots x_{i-1} x_i) = \max\{\, f(x_1 \ldots x_{i-1} y) \mid x_1 \ldots x_{i-1} y \in \mathcal{L}\,\}$. Given an objective function on $S$, an objective function $f$ on words is, respectively, a linear, bottleneck, or consistent function if there is a linear, bottleneck, or consistent function $g$ on sets such that $f(\alpha) = g(s(\alpha))$ for all words $\alpha$. This implies that if $\beta$ is a permutation of $\alpha$, then $f(\beta) = f(\alpha)$, a property often called *stability*.

The necessary and sufficient conditions for hereditary languages are (essentially) the obvious language versions of the accessible set system conditions. Consider the following language version of each our axioms. A hereditary language $(S, \mathcal{L})$ is *extensible* if it obeys:

*(extensibility axiom)* If $\alpha, \beta \in \mathcal{L}$, $\beta$ is a basic word, and $s(\alpha) \subset s(\beta)$, then $\exists x \in s(\beta) - s(\alpha)$ such that $\alpha x \in \mathcal{L}$.

A hereditary language $(S, \mathcal{L})$ is *closure-congruent* if it obeys:

*(closure-congruence axiom)* $\forall \alpha \in \mathcal{L}, \forall x, y \in ext(\alpha), \forall E \subseteq S - s(\alpha) - ext(\alpha)$, $s(\alpha) \cup \{x\} \cup E \in s^*(\mathcal{L})$ if and only if $s(\alpha) \cup \{y\} \cup E \in s^*(\mathcal{L})$.

A hereditary language, $(S, \mathcal{L})$, is *strongly extensible* if it obeys:

*(strong extensibility axiom)* If $\alpha, \beta \in \mathcal{L}$, $\beta$ is a basic word, and $|\alpha| < |\beta|$, then $\exists x \in s(\beta) - s(\alpha)$ such that $\alpha x \in \mathcal{L}$.

**Definition 4.** A *(language) matroid embedding* is a hereditary language which is extensible, closure-congruent, and the hereditary closure of which is a matroid. □

The situation regarding pathologies is more complex for hereditary languages than for accessible set systems; we say that a hereditary language is *pathological* if there exists a pair of basic words $\beta_1$ and $\beta_2$ such that $s(\beta_1) \subset s(\beta_2)$. In the language world, pathologies appear natural: if $\alpha$ and $\beta$ form a pathology, then this simply means that $\alpha$ cannot be a prefix of $\beta$.

In spite of these differences, all of our theorems hold in their obvious rephrasing.

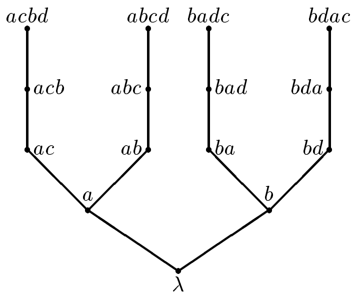
**Theorem 4.** Let $(S, \mathcal{L})$ be a hereditary language; then the following are equivalent:

1. For every positive weighted linear objective function $f$, there exists an optimal greedy basic word.
2. $(S, \mathcal{L})$ is a matroid embedding.
3. For every linear objective function, the greedy basic words are exactly the optimal basic words.

□

The generalization to the language world is not trivial, in the sense that there exist distinct language-based matroid embeddings corresponding to the same set-based matroid embedding; i.e., there exists a language-based matroid embedding which contains two feasible words that are equal as sets but have different extension sets.

**Example 4.** The language over the ground set $\{a, b, c, d\}$ with words $\lambda$, $a$, $b$, $ac$, $ab$, $ba$, $bd$, $acb$, $abc$, $bad$, $bda$, $abcd$, $acbd$, $badc$, and $bdac$, pictured below, is a matroid embedding.




Note that the words $ab$ and $ba$ are equal as sets, yet we have $ext(ab) = \{c\}$, but $ext(ba) = \{d\}$. A second language-based matroid embedding corresponding to the same set-based matroid embedding can be constructed by first building the set-based matroid embedding, then building the new language-based matroid embedding by including a feasible word for each accessibility path in the set system. □

This result should be contrasted with a theorem of Korte and Lovasz [6] showing that greedoids do not give rise to such situations.

The results presented so far for hereditary languages completely parallel those for accessible set systems. The same is almost true for the class of bottleneck functions. However, certain pathologies (that can occur in hereditary languages but not in set systems) allow the greedy algorithm to optimize all bottleneck functions on languages that fail to obey even the weaker of the extensibility axioms. By proving results paralleling Lemma 1 and Theorem 3, we can establish the following exact characterization of greedy optimality for bottleneck functions on hereditary languages.

**Theorem 5.** The best-in greedy algorithm run on a hereditary language optimizes all bottleneck functions if and only if the language is strongly extensible (except with respect to pairs of sets that form pathologies). □

# 6 Conclusion

We have presented exact characterizations of problem structures on which the greedy algorithm optimizes linear, bottleneck, and, more generally, consistent objective functions. These exact characterizations apply both to accessible set systems and hereditary languages and answer questions raised, and only partially answered, by Edmonds, Korte and Lovasz, and others.

Our results provide a framework for future research: what are additional structural properties of matroid embeddings? how can constraints about the objective function be traded against constraints on the language structure? A consequence of our results is that the linear objectives are the hardest of all consistent objectives to optimize by greedy methods on matroid embeddings, in the sense that, if they are optimized, then so too is any other consistent objective. This suggests a study of families of objective functions along much the same lines as classical complexity theory; in this direction, Lengauer and Theune [8] have demonstrated reductions among cost functions for path problems. Now that we have a proper setting for optimal greedy algorithms, we can investigate the complexity of such algorithms. This problem is harder than it may seem, since much of the structure used is given implicitly, by a feasibility oracle or some such theoretical construct: an efficient greedy algorithm results from both a fast feasibility check and a fast identification of the best extension.

# 7 References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