CS 362, HW3

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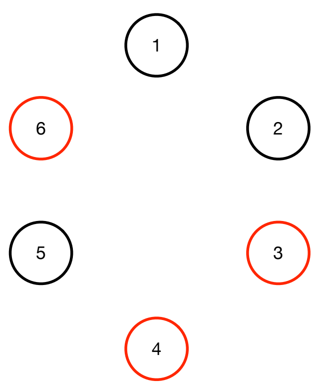
1. A cat hops on posts arranged in a circle. There are $2n$ posts, with n red and n black. The cat can start at any post, and always hops to the next post in the clockwise direction, until it visits all posts. It "wins" if, at every point during its trip, the number of red posts visited so far is always at least the number of black posts visited so far. In the figure below, the cat wins by starting at post 3, but loses if it starts at any other post.



   Prove that there is always some starting post that can ensure the cat will win. Prove this by induction on n for $n \geq 1$. Let your IH be that the cat can always win in the case where the input has at most $2(n-1)$ posts.

2. A product of matrices is *fully parenthesized* if it is either a single matrix, or the product of two fully parenthesized matrices surrounded by parenthesis. For example, $((M_1 M_2)M_3)$ and $(M_1(M_2 M_3))$ are both fully parenthesized product of 3 matrices.

   Prove by induction that any fully parenthesized product of n matrices has exactly $n-1$ pairs of parenthesis.

3. You are on an island where each inhabitant is either a knight or a knave. When asked a question, knights always tell the truth, but knaves may either lie or tell the truth. On this island, you can only ask questions of the form: "Hey Person X: what is Person Y's type?". Each person always answers a question about someone else in the same way, so there's no reason to ask the same question twice.

   (a) Assume there are n people on the island, and a strict majority of them are knights. Describe an efficient algorithm to identify the type of each person. Hint 1: You can do this in $o(n^2)$ questions. Hint 2: First try to identify 1 knight. Use recursion!

   (b) Prove that if at most half the inhabitants are knights, then it is impossible to solve the problem.

4. There are two bins: Bin 1 initially has 3 white balls and 1 red ball. Bin 2 has 4 white balls. In every round, a ball is selected uniformly at random from each bin and these two balls are swapped. Let $p(n)$ be the probability that the red ball is in bin 1 at the beginning of the n-th round.

   (a) Write a recurrence relation for $p(n)$.

   (b) (8 points) Use "guess and check", and proof by induction to solve this recurrence. Don't forget to label BC, IH and IS and clearly say where you are using the IH. Hint: Compute the first few values of $p(n)$ to spot the pattern.

   (c) If this is repeated for m rounds, what is the expected number of rounds that the red ball is in bin 1?