CS 591-06, Lecture 2

Jared Saia
University of New Mexico

Class Outline

- Review Migration Problem
- Statement of Hall's Marriage Theorem
- In-class Exercise
- Using Hall's Theorem to Factor a Graph
- Proof of Hall's Theorem

Defn: A Matching

Let G be any graph with edge set E. Then a *matching* on G is a collection of edges E' where $E' \subseteq E$ and no two edges in E' share an endpoint.

Question: If G has n vertices, what are the maximum and minimum possible sizes of a matching?

Defns

A graph G is *bipartite* if it's vertex set V can be partitioned into two sets L, R such that all edges in G have one edge in L and one edge in R.

A matching is *perfect* if for every vertex $l \in L$, there is some edge in the matching which is incident to l.

Definition: Let $G = (L, R)$. For any subset L' of L, define $N(L')$ to be the set of vertices which are neighbors of L.

Question: If $|N(L)| < |L|$, can there be a perfect matching?

Hard Question: When does a bipartite graph have a perfect matching?

Hall's Theorem

Statement of Hall's Marriage Theorem:

Let $G = (L, R)$ be a bipartite graph. Then:

- There is a perfect matching for G

if and only if

- $\forall L' \subseteq L,\ |L'| \leq |N(L')|$

We will say a bipartite graph "has Hall's property" if it fulfills this condition.

Exercise

Let $G = (L, R)$ be a k-regular bipartite graph.

Show that G *has Hall's Property (i.e.* $\forall L' \subseteq L$, $|L'| \leq |N(L')|$*)*

Euler Tour

An *Euler Tour* of a graph G is a tour which starts and ends at the same vertex and traverses every edge exactly once.

It is well known that a graph G has an Euler Tour if and only if edge vertex of the graph has even degree.

Graph Factoring

Let G be a k-regular graph where k is an even number. We factor G as follows:

1. Construct an Euler-tour of G.
2. Orient the edges according to the direction of the tour. That is, if the tour enters v on edge e_1 and leaves on edge e_2, then e_1 is an in-edge to v and e_2 is an out-edge. Thus we have $d_{\text{in}} = d_{\text{out}} = k$.
3. Set up a bipartite matching problem, B_G, with a representative of each vertex in the graph on both sides. Add in all directed edges going from left to right. Note that each edge is represented in the matching problem exactly once.
4. Find a matching (which is guaranteed to exist by Hall's Theorem). The matched edges induce a 2-factor of the original graph. Remove these edges from B_G and repeat this step until there are no edges left.

Migration Results

Let G be a graph with maximum degree Δ. Then:

Direct Migration (edge-coloring):

- Upperbound: $3\lceil\Delta/2\rceil$ steps
- Lowerbound: $2\lceil\Delta/2\rceil$ steps

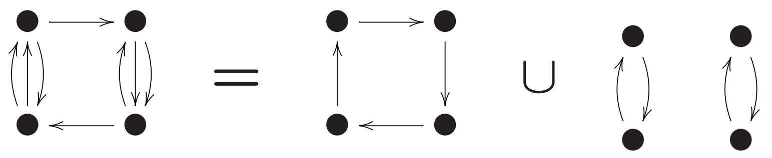
Indirect Migration (edge-coloring):

- Upperbound: $2\lceil\Delta/2\rceil$ steps with $n/3$ bypass nodes
- Lowerbound: Δ steps

Direct Migration - Results

$3\lceil \Delta/2 \rceil$ steps maximum

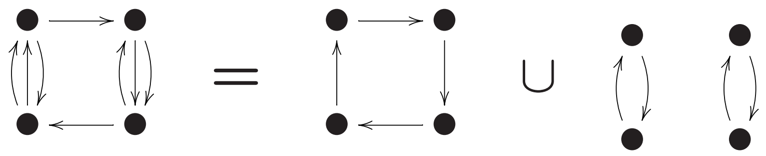
Idea: Factor graph into set of 2-regular graphs



Indirect Migration - Results

$2\lceil \Delta/2 \rceil$ steps with $n/3$ bypass nodes.

Idea: Factor graph into set of 2-regular graphs. Use $n/3$ bypass nodes for each "factor" to satisfy it in 2 steps.



Making a graph Δ'-regular

The following algorithm takes a graph with degree Δ and adds edges to it to get a graph of degree $\Delta' = 2\lceil \Delta/2 \rceil$

1. While there exists a vertex with degree less than $\Delta' - 1$, add a self loop to that vertex.
2. While there exist two distinct vertices of degree $\Delta' - 1$, add an arbitrarily directed edge between them.

Minute Paper

1. What was the most important concept you learned in this class?
2. What important question remains unanswered?
3. What was the muddiest part of the class?