MAX-CUT on Sparse Random Graphs

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August 3, 2002

Numerous NP-complete combinatorial optimization problems have been studied for random graphs. MAX-CUT, however, seems to have been left out [1]. We prove upper and lower bounds on the size of the largest cut for sparse random graphs of average degree c. While we prove explicit upper and lower bounds valid for general c, we focus on the case where c is large (but still fixed as a function of n) and prove

Theorem 1 *Let z be the size of the largest cut of a sparse random graph $G(n, p = d/n)$ with average degree c and m edges. With high probability $z = m/2 + An + o(n)$ where, for large c,*

$$\frac{1}{3}\sqrt{\frac{2}{\pi}} \leq \frac{A}{\sqrt{c}} \leq \frac{\sqrt{\ln 2}}{2} \ .$$

Our methods are standard; for the upper bound we use the first moment method, and for the lower bound we analyze a simple greedy heuristic.

We will use two different models of random graphs, $G(n, m = cn/2)$ for the upper bound and $G(n, p = c/n)$ for the lower bound. However, since the number of edges of $G(n, p)$ is tightly concentrated around its mean, with high probability (w.h.p.) the size of the largest cut differs by $o(n)$ between these two models, and so we use them interchangeably.

Proof of the upper bound. Consider a sparse random graph $G(n, m = cn/2) = (V, E)$. Let $X(z)$ be the random variable equal to the number of cuts of size z, i.e. the number of subsets $S \subset V$ such that z edges in E connect vertices in S to vertices in $V \setminus S$. We wish to bound its expectation $\mathrm{E}[X(x)]$.

*Supported by NSF PHY-0071139, the Sandia University Research Program, and Los Alamos National Laboratory.

Let $s = |S|$, and let $q = (s/n)(1-s/n)$ be the probability that a random edge is in the cut. Summing over all s we get

$$\mathrm{E}[X(z)] = \binom{m}{z} \sum_{s=0}^{n} \binom{n}{s} q^z (1-q)^{m-z} \ .$$

For $z > m/2$, $q^z(1-q)^{m-z} < 2^{-m}$, so

$$\mathrm{E}[X(z)] < 2^{n-m} \binom{m}{z} \ .$$

Let $z = m/2 + An$ and $\alpha = z/m = 1/2 + 2A/c$. Using the entropy bound on the binomial $\binom{m}{\alpha m} \leq (\alpha^\alpha (1-\alpha)^{1-\alpha})^{-m}$ and setting $m = cn/2$ we have

$$\mathrm{E}[X(z)] < \left[2 \left(\frac{1}{2\,\alpha^\alpha\,(1-\alpha)^{1-\alpha}} \right)^{c/2} \right]^n \ .$$

When the quantity inside the square brackets is less than 1, $\mathrm{E}[X(z)]$ is exponentially small, and so is the probability that a cut of size z exists. This first occurs when

$$\frac{2\ln 2}{c} = \alpha \ln \alpha + (1-\alpha)\ln(1-\alpha) + \ln 2 \ .$$

Solving this equation yields an upper bound on α for $c > 2$. When c is large, Taylor expanding the right-hand side around $\alpha = 1/2 + \epsilon$ gives

$$\frac{2\ln 2}{c} = 2\epsilon^2 + O(\epsilon^4)$$

and solving for $\epsilon = 2A/c$ gives

$$\frac{A}{\sqrt{c}} \leq \frac{\sqrt{\ln 2}}{2} + O(1/c) \ .$$

Proof of the lower bound. To get a lower bound on the size of the largest cut, we analyze a linear-time algorithm `Greedy`. Initially all vertices are uncolored. At each step, `Greedy` chooses a random vertex v and assigns to it the color opposite to the one that the majority of its neighbors have. In the case of a tie, `Greedy` chooses a color randomly. This is essentially identical to a greedy algorithm studied by Aizenman, Lebowitz and Ruelle for the Sherrington-Kirkpatrick spin glass model [2].

In the $G(n,p)$ model, v's colored neighbors are chosen independently (without replacement) from the colored vertices. After $\Theta(n)$ steps there are $\Theta(n)$ vertices colored with each color, so if v has $k = o(n)$ colored neighbors, their colors are a string of k biased coin-flips.

The worst case for `Greedy` is if an equal number of vertices are black and white, in which case this coin is fair. In fact, w.h.p. this will be the case to within $o(n)$, since the algorithm clearly possesses a "dynamical stability" which redresses any imbalance between the two colors.

Therefore, let $m(k)$ be the expected size of the majority of k coin-flips. To calculate $m(k)$, consider flipping the k coins in order. It is easy to see that $m(k)$ increases by $1/2$ whenever there is currently a majority (since the next coin flip either increases the majority or leaves it unchanged), and by 1 whenever the heads and tails are currently equal (since whichever becomes the majority will now be larger). This gives

$$\begin{aligned} m(k) &= \frac{k}{2} + \frac{1}{2} \sum_{j=0,2,4,\ldots}^{k-1} 2^{-j} \binom{j}{j/2} \\ &= \frac{k}{2} + \frac{k'}{2} 2^{-k'} \binom{k'}{k'/2} \end{aligned}$$

where $k' = 2\lceil k/2 \rceil$ is the smallest even number greater than or equal to k.

Now suppose that `Greedy` is on its t'th step. In that case k will be a random variable which is Poisson distributed with mean ct/n,

$$P_{ct/n}(k) = \mathrm{e}^{-ct/n} \frac{(ct/n)^k}{k!}$$

since in $G(n,p)$ each edge between v and the colored vertices appears independently. Then on this step `Greedy` will increase the size of the cut by the expectation of $m(k)$,

$$\begin{aligned} \mathrm{E}[m(k)] &= \sum_{k=0}^{\infty} P_{ct/n}(k)\, m(k) \\ &= \frac{ct}{2n} \left(1 + \mathrm{e}^{-ct/n} \big(I_0(ct/n) + I_1(ct/n)\big)\right) \end{aligned}$$

where I_0 and I_1 are modified Bessel functions of the first kind.

By linearity of expectation, `Greedy` finds a cut whose expected size is $\sum_{t=0}^{n-1} \mathrm{E}[m(k)]$. Since the terms in this sum increase monotonically, we can replace it by an integral with an additive error equal to its largest term, which is $O(1)$. Thus we have a lower bound on the size of the largest cut, up to $o(n)$, of

$$\begin{aligned} z &\geq \int_0^n \mathrm{E}[m(k)]\, \mathrm{d}t \\ &= \frac{cn}{4} + \frac{\mathrm{e}^{-c} n}{6} \big(2cI_0(c) + (2c-1)I_1(c)\big) \end{aligned}$$

(Note that $cn/4 = m/2$.)

When c is large, both $I_0(c)$ and $I_1(c)$ can be approximated by $\mathrm{e}^c/\sqrt{2\pi c}\,(1 + O(1/c))$, giving us

$$\frac{A}{\sqrt{c}} \geq \frac{1}{3}\sqrt{\frac{2}{\pi}} + O(1/c) \ .$$

Of course, this can be improved for small c, e.g. by 2-coloring the trees outside the giant component.

We conclude by noting that our results can also be interpreted as $\Theta(n\sqrt{c})$ bounds on the ground-state energy of the antiferromagnetic Ising model defined on sparse random graphs, which in our notation is $-2An$.

References

[1] Frieze, A. and McDiarmid, C., "Algorithmic theory of random graphs." *Random Structures and Algorithms* **10** (1997) 5–42.

[2] Aizenman, M., Lebowitz, J. L., Ruelle, D., "Some rigorous results on the Sherrington-Kirkpatrick spin glass model." *Communications in Mathematical Physics* **112** (1987) 3–20.