Distributed Algorithms

Jared Saia
Ouisa Kitteridge: “I read somewhere that everybody on this planet is separated by only six other people. Six degrees of separation between us and everyone else on this planet. The President of the United States, a gondolier in Venice, just fill in the names. I find it extremely comforting that we're so close. I also find it like Chinese water torture, that we're so close because you have to find the right six people to make the right connection.”
**6 Degrees**

**Ouisa Kitteridge**: “I read somewhere that everybody on this planet is separated by only six other people. Six degrees of separation between us and everyone else on this planet. The President of the United States, a gondolier in Venice, just fill in the names. I find it extremely comforting that we're so close. I also find it like Chinese water torture, that we're so close because you have to find the right six people to make the right connection.”

**Tess**: “He offered you parts in Cats? I thought you hated Cats. You said it was an all time low in a lifetime of theatre going. You said, "Aeschylus did not invent the theatre to have it end up a bunch of chorus kids in cat suits prancing around wondering which of them will go to kitty-cat heaven."
**Milgram’s Experiment**

Start: 160 random people in Omaha
Target: 1 stock broker in Boston

Rule: Only send to a friend or acquaintance
Milgram’s Experiment

Start: 160 random people in Omaha
Target: 1 stock broker in Boston

Rule: Only send to a friend or acquaintance

Result: takes 6 hops on average to get to target
Milgram’s Experiment

Start: 160 random people in Omaha
Target: 1 stock broker in Boston

Rule: Only send to a friend or acquaintance

Result: takes 6 hops on average to get to target

Recent: ~6 hops to route via email (Watts, ‘01)
Social Network Properties

1) Shortest paths are small

"Six degrees of separation ... I find it extremely comforting that we're so close."

2) Local Clusters

"Keep your friends close and your enemies closer" - Machiavelli
Watts-Strogatz Model

“Small World” model ensures both:

- Short paths (logarithmic)
- Many clusters
Watts-Strogatz Model

- "Small World" model ensures both:
  - Short paths (logarithmic)
  - Many clusters

Small World is ordered + random
Figure 20-3: The general conclusions of the Watts-Strogatz model still follow even if only a small fraction of the nodes on the grid each have a single random link—two nodes are one grid step apart if they are directly adjacent to each other in either the horizontal or vertical direction—We now create a network by giving each node two kinds of links: those explainable purely by homophily and those that constitute weak ties—Homophily is captured by having each node form a link to all other nodes that lie within a radius of up to \( r \) grid steps away for some constant value of \( r \): these are the links you form to people because you are similar to them—Then each node also forms a link to \( k \) other nodes selected uniformly at random from the grid—these correspond to weak ties connecting nodes who lie very far apart on the grid—Figure 20-2ubv gives a schematic picture of the resulting network—a hybrid structure consisting of a small amount of randomness the weak ties sprinkled onto an underlying structured pattern the homophilous links—Watts and Strogatz observe first that the network has many triangles: any two neighboring nodes or nearby nodes will have many common friends where their neighborhoods of radius \( r \) overlap and this produces many triangles—But they also find that there are—with high probability—very short paths connecting every pair of nodes in the network—Roughly the argument is as follows—Suppose

1) ordered links: neighbors in grid
2) random links: to random node in grid
Each node has one random link
Watts-Strogatz

1) ordered links: neighbors in grid
2) random links: to random node in grid

Each node has one random link

Clear that: 1) Many local clusters;
Can show: 2) All distances at most logarithmic.
Watts-Strogatz

1) ordered links: neighbors in grid
2) random links: to random node in grid
Each node has one random link

Clear that: 1) Many local clusters;
Can show: 2) All distances at most logarithmic.
A Problem

“Six degrees of separation ... I find it extremely comforting that we're so close... I also find it like Chinese water torture, that we're so close because you have to find the right six people to make the right connection.”

Knowing there exist six people is very different than finding those six people
A Problem

“Six degrees of separation ... I find it extremely comforting that we're so close... I also find it like Chinese water torture, that we're so close because you have to find the right six people to make the right connection.”

Knowing there exist six people is very different than finding those six people.

In fact, Watts-Strogatz is wrong! It doesn’t account for finding the six people.
The general conclusions of the Watts-Strogatz model still follow even if only a small fraction of the nodes on the grid each have a single random link—two nodes are one grid step apart if they are directly adjacent to each other in either the horizontal or vertical direction. We now create a network by giving each node two kinds of links: those explainable purely by homophily and those that constitute weak ties. Homophily is captured by having each node form a link to all other nodes that lie within a radius of up to $r$ grid steps away; these are the links you form to people because you are similar to them. Then, for some other constant value $k$, each node also forms a link to $k$ other nodes selected uniformly at random from the grid—these correspond to weak ties connecting nodes who lie very far apart on the grid. Figure 20-2ubv gives a schematic picture of the resulting network—a hybrid structure consisting of a small amount of randomness—the weak ties—sprinkled onto an underlying structured pattern—the homophilous links. Watts and Strogatz observe first that the network has many triangles: any two neighboring nodes or nearby nodes will have many common friends where their neighborhoods of radius $r$ overlap and this produces many triangles. But they also find that there are—with high probability—very short paths connecting every pair of nodes in the network. Roughly, the argument is as follows: Suppose start target $n$ nodes in grid

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A Problem

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Figure 20-2 gives a schematic picture of the resulting network: a hybrid structure consisting of a small amount of randomness (the weak ties) sprinkled onto an underlying structured pattern (the homophilous links).

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Figure 20-3: The general conclusions of the Watts-Strogatz model still follow even if only a small fraction of the nodes on the grid each have a single random link—two nodes are one grid step apart if they are directly adjacent to each other in either the horizontal or vertical direction.

Q: What is expected time to get to this red square?

n nodes in grid

target

Region containing \( \sqrt{n} \) nodes

n nodes in grid

start

\( \sqrt{n} \) nodes

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Figure 20-3: The general conclusions of the Watts-Strogatz model still follow even if only a small fraction of the nodes on the grid each have a single random link—two nodes are one grid step apart if they are directly adjacent to each other in either the horizontal or vertical direction.

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A Problem

Q: What is expected time to get to this red square?

Using short links alone requires $\sqrt{n}$ hops
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Using short links alone requires $\sqrt{n}$ hops

A long link has prob. $1/\sqrt{n}$ of falling in red square
A Problem

Q: What is expected time to get to this red square?

Using short links alone requires $\sqrt{n}$ hops

A long link has prob. $1/\sqrt{n}$ of falling in red square

Expect to have to visit $\sqrt{n}$ nodes before finding a long link which falls in red square!
A Problem

Expect to have to visit \( \sqrt{n} \) nodes before finding a long link which falls in red square!
A Problem

Expect to have to visit $\sqrt{n}$ nodes before finding a long link which falls in red square!

307 million people in the United States
A Problem

Expect to have to visit $\sqrt{n}$ nodes before finding a long link which falls in red square!

307 million people in the United States

$\sqrt{307}$ million is about 17,500
A Problem

Expect to have to visit $\sqrt{n}$ nodes before finding a long link which falls in red square!

307 million people in the United States

$\sqrt{307}$ million is about 17,500

Need much quicker routing!!!
Kleinberg Model

1) ordered links: neighbors in grid
2) random links: to random node in grid
Each node has one random link

Watts-Strogatz:
Node selected uniformly at random
Kleinberg Model

1) ordered links: neighbors in grid
2) random links: to random node in grid

Each node has one random link

Watts-Strogatz: Node selected uniformly at random

Kleinberg: Node x selected with probability
\[ \propto \frac{1}{(\text{distance to } x)^2} \]
CHAPTER 20. THE SMALL-WORLD PHENOMENON

(a) A small clustering exponent

(b) A large clustering exponent

Figure sqov— With a small clustering exponent, the random edges tend to span long distances on the grid; as the clustering exponent increases, the random edges become shorter.

We will evaluate different search procedures according to their delivery time—the expected number of steps required to reach the target over a randomly generated set of long-range contacts and randomly chosen starting and target nodes.

Unfortunately, given this setup, one can prove that decentralized search in the Watts-Strogatz model will necessarily require a large number of steps to reach a target—much larger than the true length of the shortest path [suw]. As a mathematical model, the Watts-Strogatz network is thus effective at capturing the density of triangles and the existence of short paths, but not the ability of people working together in the network to actually find the path. Essentially, the problem is that the weak ties that make the world small are “too random” in this model—since they’re completely unrelated to the similarity among nodes that produces the homophily-based links, they’re hard for people to use reliably.

One way to think about this is in terms of Figure sqou, a hand-drawn image from Milgram’s original article in Psychology Today. In order to reach a faraway target, one must use long-range weak ties in a fairly structured, methodical way, constantly reducing the distance to the target. As Milgram observed in the discussion accompanying this picture, “The geographic movement of the [letter] from Nebraska to Massachusetts is striking. There is a progressive closing in on the target area as each new person is added to the chain” [suy]. So it is not enough to have a network model in which weak ties span only the very long ranges; it is necessary to span all the intermediate ranges of scale as well. Is there a simple way to adapt the model to take this into account?

1) ordered links: neighbors in grid
2) random links: to random node in grid
Each node has one random link

Watts-Strogatz: Node selected uniformly at random

Kleinberg: Node x selected with probability \( \propto \frac{1}{(\text{distance to } x)^2} \)
Kleinberg

- Result: In Kleinberg model, can route from any start node to any goal node in essentially $\log^2 n$ hops!
CHAPTER 20. THE SMALL-WORLD PHENOMENON

Empirical Analysis and Generalized Models

The results we've seen thus far have been for stylized models but they raise a number of qualitative issues that one can try corroborating with data from real social networks. In this section we discuss empirical studies that analyze geographic data to look for evidence of the exponent $q = 2$ as well as more general versions of these models that incorporate non-geographic notions of social distance.

Geographic Data on Friendship.

In the past few years, the rich data available on social networking sites has made it much easier to get large-scale data that provides insight into how friendship links scale with distance. Liben-Nowell et al. [2005] used the blogging site LiveJournal for precisely this purpose, analyzing roughly 15,000 users who provided a ZIP code for their home address, as well as links to their friends on the system. Note that LiveJournal is serving here primarily as a very useful "model system," containing data on the geographic basis of friendship links on a scale that would be enormously difficult to obtain by more traditional survey methods.

From a methodological point of view, it is an interesting and fairly unresolved issue to understand how closely the structure of friendships defined in online communities corresponds to the structure of friendships as we understand them in offline settings.

A number of things have to be done in order to align the LiveJournal data with the basic grid model, and perhaps the most subtle involves the fact that the population density of the LiveJournal network is extremely nonuniform as it is for the United States as a whole. See Figure 20.8 for a visualization of the population density in the LiveJournal data. In particular, the rank addresses variations in population density.

Data

Population density of LiveJournal network
(Liben-Nowell et al. '05)

Rank addresses variations in population density
Data

Population density of LiveJournal network (Liben-Nowell et al. ’05)

Rank addresses variations in population density

General Case: prob. of link to node w/ rank r is $\propto 1/r$
Rank addresses variations in population density.

General Case: prob. of link to node \( w \) / rank \( r \) is

\[ \propto 1/r \]

Acknowledgement: Many of the figures in this talk are from the book *Networks, Crowds and Markets: Reasoning about a Highly Connected World* by David Easley and Jon Kleinberg.
CHAPTER 20. THE SMALL-WORLD PHENOMENON

(a) Rank-based friendship on LiveJournal

(b) Rank-based friendship: East and West coasts

Observed probability fits very close to $1/r$
20.7 Advanced Material: Analysis of Decentralized Search

In Section 20.4, we gave some basic intuition for why an inverse-square distribution of links with distance makes effective decentralized search possible. Even given this way of thinking about it, however, it still requires further work to really see why search succeeds with this distribution. In this section, we describe the complete analysis of the process [249].

To make the calculations a bit simpler, we vary the model in one respect: we place the nodes in one dimension rather than two. In fact, the argument is essentially the same no matter how many dimensions the nodes are in, but one dimension makes things the cleanest even if not the best match for the actual geographic structure of a real population. It turns out, as we will argue more generally later in this section, that the best exponent for search is equal to the dimension, so in our one-dimensional analysis we will be using an exponent of $q = -1$ rather than $q = 2$. At the end, we will discuss the minor ways in which the argument needs to be adapted in two or higher dimensions.

We should also mention, recalling the discussion earlier in the chapter, that there is a second fundamental part of this analysis as well — showing that this choice of $q$ is in fact the best for decentralized search in the limit of increasing network size. At the end, we sketch why this is true, but the full details are beyond what we will cover here.

Easier to do analysis on a ring (but same techniques work for a grid)

Random link to $x$ will now happen with probability $\propto 1/(\text{distance to } x)$
Algorithm: Current message holder forwards message to person it knows who is closest to target.
We’ll say we’re in phase j of the algorithm when distance from target is between $2^j$ and $2^{j-1}$.
Analysis

We’ll say we’re in phase $j$ of the algorithm when distance from target is between $2^j$ and $2^{j-1}$.

Number of phases is $\log n$. 
We’ll say we’re in phase $j$ of the algorithm when distance from target is between $2^j$ and $2^{j-1}$

Number of phases is $\log n$

Let $X = \#$ hops total; $X_i = \#$ hops in phase $i$
Analysis

We’ll say we’re in phase $j$ of the algorithm when distance from target is between $2^j$ and $2^{j-1}$

Number of phases is $\log n$

Let $X = \#$ hops total; $X_i = \#$ hops in phase $i$

Then $X = X_1 + X_2 + ... + X_{\log n}$
Analysis

Then $X = X_1 + X_2 + \ldots + X_{\log n}$
Analysis

Then $X = X_1 + X_2 + \ldots + X_{\log n}$

$E(X) = E(X_1 + X_2 + \ldots + X_{\log n})$
Analysis

Then \( X = X_1 + X_2 + ... + X_{\log n} \)

\[
E(X) = E(X_1 + X_2 + ... + X_{\log n})
\]

\[
E(X) = E(X_1) + E(X_2) + ... + E(X_{\log n})
\]

By Linearity of Expectation!
Analysis

\[ E(X) = E(X_1) + E(X_2) + \ldots + E(X_{\log n}) \]

Now we “just” need to calculate \( E(X_i) \), the expected number of hops in phase \( i \).

To do this, we calculate the probability that a single random link allows us to end phase \( i \).
Probabilities

Recall: Random link to from u to v occurs with probability
\( \propto \frac{1}{\text{distance from u to v}} \)

Normalizing constant \( Z \) is the sum over all v of \( \frac{1}{\text{distance from u to v}} \)

\[ Z \leq 2\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots \frac{1}{(n/2)}\right) \]
\[ Z \leq 2\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{(n/2)}\right) \]
\[ Z \leq 2 \left( 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n/2} \right) \]

But:

\[ (1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots \frac{1}{k}) \leq 1 + \int_{1}^{k} \frac{1}{x} \, dx \]
\[ Z \leq 2 \left( 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{(n/2)} \right) \]

But:

\[ \left( 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{k} \right) \leq 1 + \int_{1}^{k} \frac{1}{x} \, dx \]

And:

\[ 1 + \int_{1}^{k} \frac{1}{x} \, dx = 1 + \ln k \]
\[ Z \leq 2 \left( 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n/2} \right) \]

But:
\[ (1 + 1/2 + 1/3 + 1/4 + \ldots + 1/k) \leq 1 + \int_{1}^{k} \frac{1}{x} \, dx \]

And:
\[ 1 + \int_{1}^{k} \frac{1}{x} \, dx = 1 + \ln k \]

So:
\[ Z \leq 2 \left( 1 + \ln(n/2) \right) \leq 2 \log_{2} n \]
Probabilities

So:

\[ Z \leq 2\left(1 + \ln\left(\frac{n}{2}\right)\right) \leq 2 \log_2 n \]
So:

\[ Z \leq 2(1 + \ln(n/2)) \leq 2 \log_2 n \]

Let \( d(u,v) = \) distance from \( u \) to \( v \). Then prob. \( u \) links to \( v \) is

\[ \frac{1}{Z} d(u, v)^{-1} \geq \frac{1}{2 \log n} d(u, v)^{-1} \]

Only remaining task is to add up these probabilities over all vertices \( v \) that will let us exit the current phase.
d+1 nodes within distance d/2 of t
d+1 nodes within distance d/2 of t

Prob. of hitting particular node v in there at least:

\[ \frac{1}{2 \log n} d(u, v)^{-1} \geq \frac{1}{2 \log n} \frac{1}{3d/2} = \frac{1}{3d \log n} \]
d+1 nodes within distance d/2 of t

Prob. of hitting particular node v in there at least:

\[ \frac{1}{2 \log n} d(u, v)^{-1} \geq \frac{1}{2 \log n} \frac{1}{3d/2} = \frac{1}{3d \log n} \]

\( d+1 \) total nodes; prob. of hitting one is at least:

\[ (d + 1) \frac{1}{3d \log n} = \frac{1}{3 \log n} \]
Phases

So we’re walking around in phase $j$

Every time we see a random edge, it has prob. at least $1/(3 \log n)$ of taking us to next phase

Q: How long do we expect to walk before finding one of these special edges?
Phases

Q: How long do we expect to walk before finding one of these special edges?

Q: If a coin has probability $p$ of coming up heads, how many times do you expect to flip it before you get heads?

A: $1/p$
Phases

Q: How long do we expect to walk before finding one of these special edges?

Q: If a coin has probability $p$ of coming up heads, how many times do you expect to flip it before you get heads?

A: $1/p$

$$E(X) = p*1 + (1-p)(1 + E(X))$$

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Wrapup

Recall: \( E(X) = E(X_1) + E(X_2) + \ldots + E(X_{\log n}) \)

Thus: \( E(X) \leq 3 \log n + 3 \log n + \ldots + 3 \log n \)

\( \leq 3 \log^2 n \)
Recall: $E(X) = E(X_1) + E(X_2) + \ldots + E(X_{\log n})$

Thus: $E(X) \leq 3 \log n + 3 \log n + \ldots + 3 \log n$

$\leq 3 \log^2 n$

The End!
Wrapup

Recall: $E(X) = E(X_1) + E(X_2) + ... + E(X_{\log n})$

Thus: $E(X) \leq 3 \log n + 3 \log n + ... + 3 \log n$

$\leq 3 \log^2 n$

The End!

Or is It???
Open Questions

- Why do friendship links have the Kleinberg exponent?
- Why should routing speed determine the way in which we make friends?
- Why do we have friends?
Graph Coloring

- Must color each node in a graph (network)
- A coloring is **valid** if any pair of nodes that are linked have different colors
- Goal: Find a valid coloring using the smallest number of colors
Graph Coloring

Example graph and valid 3 coloring
Graph Coloring

- Unlike shortest paths, coloring is computational hard even when centralized
- Sudoku is a graph coloring problem (with some colors already fixed)
- How?
Distributed Coloring

- Division of resources in social networks
- Nodes are people, links represent friendships
- Colors are resources
- Goal: Assign resources to people so that friends don’t fight over the same resource
- Distributed: Each node knows only local neighborhood
Example Resources

- **Time**: scheduling talks in conference rooms
- **Economic**: pursuing different expertise/markets by people/companies
- **Political**: pursuing different political offices
- **Technological**: selecting a channel unused by close parties in a wireless network
An Experiment

Kearns et al. ’06 ran a distributed coloring experiment on people
Graphs Used

"Small World" (Watts-Strogatz)

Kleinberg?!?

Preferential attachment

We have been performing human subject experiments in a variety of simple and complex networks. Subjects each simultaneously interact with network dynamics (as induced by geography) with long-range distance connectivity (as induced by travel or chance meetings) often found in social and organizational networks (developing an expertise not difficult by parties in a wireless communication network or by others nearby). Graph coloring also distinguishes one behavior from that of one's network neighbors. Networks generated by preferential attachment made solving the graph coloring problem, which models settings in which it is desirable to distinguish one's behavior from that of one's network neighbors. Networks generated by preferential attachment made solving the graph coloring problem, which models settings in which it is desirable to distinguish one's behavior from that of one's network neighbors.

The graph coloring problem is a natural problem faced by faculty members scheduling departmental events—recurring classes, one-time seminars, exams, and so on—in a limited number of available rooms. We can view the events to be connected by a graph, with an edge between any two events that temporally overlap, even partially. Clearly, two such events cannot both happen simultaneously, and there are two possibilities: either the events are separate, in which case we are done, or they overlap, in which case we are faced with the necessity of choosing a different room for one or both of the events. Thus we have a problem of coloring a graph with a minimal number of colors, subject to the condition that no two adjacent vertices be assigned the same color.

The coloring problem was chosen for both its simplicity of description and its contrast to other coordination problems. Unlike the well-studied navigation or shortest-paths problem, optimal coloring is notoriously intractable from the viewpoint of even weak approximations (in which many more colors than the chromatic number are permitted) generalize many traditional problems in logistics and operations research (developing an expertise not difficult by parties in a wireless communication network or by others nearby). Graph coloring also distinguishes one behavior from that of one's network neighbors. Networks generated by preferential attachment made solving the graph coloring problem, which models settings in which it is desirable to distinguish one's behavior from that of one's network neighbors.

We report here on the findings from two series of coloring experiments in which the subjects were asked to color graphs used in the like. Other coloring-like problems arise in a variety of social activities (such as selecting a cell phone ringtone that differs from those of family members, friends, and colleagues); technological coordination (selecting a channel unused by near-neighbor networks when setting up a wireless communication network); and a semantic notion that is inordinately important in the routing of information in social and organizational networks.

Theoretical work suggests that structural properties of naturally occurring networks are important in controlling a single vertex in a network of 38 vertices. Resolution of these properties in networks from many different worlds’ networks were easier still. We also showed that providing more information can have dramatically different effects in different settings. Theoretical models have confirmed earlier empirical appearances of certain structural properties in networks from many different worlds’ networks were easier still. We also showed that providing more information can have dramatically different effects in different settings.

A small world is a network with a short average path length and high clustering. Small worlds were first observed in social networks, and have since been found in many other types of networks, including technological and biological networks. The term "small world" was coined by Stanley Milgram in the 1960s to describe a type of network in which the average distance between any two nodes is relatively small. Small worlds have been studied extensively in the fields of sociology, psychology, and computer science, and have been used to model a wide range of phenomena, from the spread of diseases to the structure of the internet.

Small worlds are characterized by a high clustering coefficient, which is the fraction of pairwise neighbors of a vertex that are also connected to each other. The clustering coefficient is a measure of the "cliquishness" of a network, and small worlds are known for their high clustering coefficient. Additionally, small worlds have a high degree of path length, which is the average number of steps required to connect any two nodes in the network. Small worlds are known for their high degree of path length, which is a measure of the "smallness" of the network.

Small worlds are often modeled using the Watts-Strogatz model, which is a widely used model of small-world networks. The Watts-Strogatz model is based on a regular lattice, where each node is connected to its nearest neighbors. Small-world properties are achieved by randomly rewiring some of the edges with a probability p. The parameter p controls the degree of small-worldness, with higher values of p leading to more random networks and lower values of p leading to more regular networks.

Another model of small-world networks is the Barabási-Albert model, which is based on the preferential attachment principle. In this model, new nodes are added to the network over time, and are connected to pre-existing nodes with a probability proportional to the degree of the node. This leads to a scale-free network, where the degree distribution follows a power law. Scale-free networks are known for their high degree of path length and high clustering coefficient, and are often used to model real-world networks such as the internet and social networks.

Small worlds are important in many fields, including sociology, psychology, and computer science. They have been used to model a wide range of phenomena, from the spread of diseases to the structure of the internet. Small worlds are characterized by a high clustering coefficient and a high degree of path length, and are often modeled using the Watts-Strogatz or Barabási-Albert models.
Empirical Results

<table>
<thead>
<tr>
<th>Graph statistics</th>
<th>Colors required (No.)</th>
<th>Min. links (No.)</th>
<th>Max. links (No.)</th>
<th>Avg. links (No.)</th>
<th>SD</th>
<th>Avg. distance (No. of links)</th>
<th>Avg. experiment duration (s) and fraction solved</th>
<th>Distributed heuristic (No. of color changes)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Simple cycle</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>9.76</td>
<td>144.17</td>
<td>378</td>
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<tr>
<td>5-chord cycle</td>
<td>2</td>
<td>2</td>
<td>4</td>
<td>2.26</td>
<td>0.60</td>
<td>5.63</td>
<td>121.14</td>
<td>687</td>
</tr>
<tr>
<td>20-chord cycle</td>
<td>2</td>
<td>2</td>
<td>7</td>
<td>3.05</td>
<td>1.01</td>
<td>3.34</td>
<td>65.67</td>
<td>8265</td>
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<tr>
<td>Leader cycle</td>
<td>2</td>
<td>3</td>
<td>19</td>
<td>3.84</td>
<td>3.62</td>
<td>2.31</td>
<td>40.86</td>
<td>8797</td>
</tr>
<tr>
<td>Pref. att., v = 2</td>
<td>3</td>
<td>2</td>
<td>13</td>
<td>3.84</td>
<td>2.44</td>
<td>2.63</td>
<td>219.67</td>
<td>1744</td>
</tr>
<tr>
<td>Pref. att., v = 3</td>
<td>4</td>
<td>3</td>
<td>22</td>
<td>5.68</td>
<td>4.22</td>
<td>2.08</td>
<td>154.83</td>
<td>4703</td>
</tr>
</tbody>
</table>

Small world easy
Preferential attachment hard
Maximal Independent Set

To solve distributed graph coloring, we first address a simpler problem:

**Independent Set:** A set of nodes in a network, such that there is no edge between any pair in the set.

An independent set is *maximal* if no nodes can be added.
Maximal Independent Set

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Acknowledgement: Much of the discussion here is based on lecture notes by Roger Wattenhofer at [http://www.dcg.ethz.ch/lectures/podc/](http://www.dcg.ethz.ch/lectures/podc/)
Maximal Independent Set

This is a maximal independent set
Maximal Independent Set

This is a maximal independent set
Note: all nodes in an independent set can be colored with the same color

Friday, June 24, 2011
An MIS Algorithm
An MIS Algorithm

1) Each node \( v \) chooses a random value, \( r(v) \), in \([0,1]\) and sends it to its neighbors.
An MIS Algorithm

1) Each node $v$ chooses a random value, $r(v)$, in $[0,1]$ and sends it to its neighbors.

2) If $r(v) < r(w)$ for all neighbors $w$ of $v$, then $v$ enters the MIS and informs its neighbors.
An MIS Algorithm

1) Each node v chooses a random value, r(v), in [0,1] and sends it to its neighbors.
2) If r(v) < r(w) for all neighbors w of v, then v enters the MIS and informs its neighbors.
3) If v or a neighbor entered the MIS, it terminates (removing all edges); otherwise go back to step 1.
Some Facts

- The algorithm always finds a MIS
- The algorithm terminates since in each loop, at least one node is added
- Q: How fast is the algorithm?
Analysis

- We’ll show that, in expectation, half of the edges are removed in each loop of the algorithms.
- This implies that number of loops is only $\log m$ where $m$ is number of edges, $n$ number of nodes.
- Since $m \leq n^2$, we know that $\log m \leq 2 \log n$.
- We’ll let $d(x)$ be the “degree of x” i.e. number of edges incident to $x$. 
A Clever Trick

Let $v \Rightarrow w$ be the event that $r(v) \leq r(w)$ and $r(v) \leq r(x)$ for all neighbors $x$ of $w$.
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Note that $X \leq (1/2) \times$ total number of edges removed!
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Let $X = \sum_{((v,w) \in E)} X_{v \Rightarrow w}$, where $E$ is the set of edges.

Note that $X \leq (1/2) \times \text{total number of edges removed!}$

Since for any edge $(s,t)$, at most one event $X^* \Rightarrow s$ and at most one event $X^* \Rightarrow t$ can happen.
A Clever Trick

Now all that remains is to compute $E(X)$

$$E(X) = E\left(\sum_{(v,w) \in E} X_{v \rightarrow w}\right)$$
A Clever Trick

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\[
E(X) = E\left(\sum ((v,w) \text{ in } E) \ X_{v\Rightarrow w}\right)
\]

\[
E(X) = \sum ((v,w) \text{ in } E) \ E(X_{v\Rightarrow w}) + E(X_{w\Rightarrow v})
\]
A Clever Trick

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$$E(X) = E\left(\sum_{(v,w) \in E} X_{v \Rightarrow w}\right)$$

$$E(X) = \sum_{(v,w) \in E} E(X_{v \Rightarrow w}) + E(X_{w \Rightarrow v})$$

$$= \sum_{(v,w) \in E} \Pr(\text{event } v \Rightarrow w) \cdot d(w) + \Pr(\text{event } w \Rightarrow v) \cdot d(v)$$
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$$\geq \sum_{(v,w) \in E} \frac{d(w)}{d(v)+d(w)} + \frac{d(v)}{d(w)+d(v)}$$
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Now all that remains is to compute $E(X)$

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$$\geq \sum_{(v,w) \in E} d(w)/(d(v)+d(w)) + d(v)/(d(w) + d(v))$$

$$= \sum_{(v,w) \in E} 1$$
A Clever Trick

Now all that remains is to compute $E(X)$

$$E(X) = E(\sum ((v,w) \in E) \ X_{v \Rightarrow w})$$

$$E(X) = \sum ((v,w) \in E) \ E(X_{v \Rightarrow w}) + E(X_{w \Rightarrow v})$$

$$= \sum ((v,w) \in E) \ Pr(\text{event } v \Rightarrow w) \ d(w) + Pr(\text{event } w \Rightarrow v) d(v)$$

$$\geq \sum ((v,w) \in E) \ d(w) / (d(v) + d(w)) + d(v) / (d(w) + d(v))$$

$$= \sum ((v,w) \in E) \ 1$$

$$= m$$
Recap

- We’ve shown that $E(X) = m$
- We also shown that the number of edges removed in each loop is at least $X/2$
- Implies that we expect half the edges to be removed in each loop
- Thus, we expect only $\log m$ iterations of the loop!
Recap

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The End!

Or is It???
Open Problems

- A major open problem in distributed computing is whether or not we can do better than logarithmic time for MIS.

- Or at least come up with a deterministic algorithm that takes logarithmic time.
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- Or at least come up with a deterministic algorithm that takes logarithmic time.

Also: hey, what about graph coloring?
1) Each node $v$ makes $d(v)+1$ clones. All clones of $v$ are linked together.

2) If $u$ and $v$ neighbors, then for all $i$, the $i$-th clone of $u$ is linked to the $i$-th clone of $v$. 
Create New Graph

1) Each node $v$ makes $d(v)+1$ clones. All clones of $v$ are linked together.
2) If $u$ and $v$ neighbors, then for all $i$, the $i$-th clone of $u$ is linked to the $i$-th clone of $v$.

Only links between 1st clones shown.
1) Each node $v$ makes $d(v)+1$ clones. All clones of $v$ are linked together.

2) If $u$ and $v$ neighbors, then for all $i$, the $i$-th clone of $u$ is linked to the $i$-th clone of $v$.

3) We now run the MIS algorithm on the new graph. If the $i$-th clone of $v$ is in the MIS, $v$ is colored $i$!
A Coloring Algorithm
Fact 1: For any node v, at most one clone is in the MIS
A Coloring Algorithm

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Fact 2: For any node v, at least one clone is in the MIS
Fact 3: The running time is logarithmic since the new graph has at most $m^2$ edges
Wrapup
The algorithm colors any graph with $\Delta + 1$ colors, where $\Delta$ is the maximum degree of a node.
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The algorithm takes time logarithmic in $n$. 
Wrapup

The algorithm colors any graph with $\Delta+1$ colors, where $\Delta$ is the maximum degree of a node.

The algorithm takes time logarithmic in $n$.

Note: $\Delta$ is not necessarily the minimum number of colors needed!
Questions
Note: There are faster coloring algorithms (log log n is even possible)
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Question: How does the structure of the graph (small world, preferential attachment) effect the difficulty of graph coloring in practice?

Answer: We don’t know!
Conclusion

- Many problems can be solved efficiently over large networks
- Randomness is a powerful tool, but need to get the distributions right!
- Interaction between Form (topology) and Function (computation) is critical
- Still much work needed to understand this interaction