7.3 Closure properties of the regular languages

Stephan Falke

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In this section we are going to show that the class of regular languages is closed under complementation, union, and intersection.

Definition 7.3.1
Let L_1 and L_2 be languages over the alphabet Σ.

- COMPLEMENTATION: $\overline{L_1} := \Sigma^* - L_1$
- UNION: $L_1 \cup L_2 := \{x \in \Sigma^* \mid x \in L_1 \text{ or } x \in L_2\}$
- INTERSECTION: $L_1 \cap L_2 := \{x \in \Sigma^* \mid x \in L_1 \text{ and } x \in L_2\}$

Proposition 7.3.2
If L is a regular language, then so is $\overline{L}$.

Proof. Let L be a regular language, and let $A = (Q, \Sigma, q_0, \delta, F)$ be a DFA that accepts L. Consider the DFA $\overline{A} = (Q, \Sigma, q_0, \delta, Q - F)$.

Then

$$\begin{aligned}
L(\overline{A}) &= \{x \in \Sigma^* \mid \delta^*(q_0, x) \in Q - F\} \\
&= \{x \in \Sigma^* \mid \delta^*(q_0, x) \notin F\} \\
&= \Sigma^* - \{x \in \Sigma^* \mid \delta^*(q_0, x) \in F\} \\
&= \Sigma^* - L(A) \\
&= \Sigma^* - L \\
&= \overline{L}
\end{aligned}$$

Hence, the DFA $\overline{A}$ accepts the language $\overline{L}$, which is thus regular. □

Proposition 7.3.3
If L_1 and L_2 are regular languages, then so is $L_1 \cap L_2$.

Proof. Let L_i be regular and accepted by the DFA $A_i = (Q_i, \Sigma, q_{0,i}, \delta_i, F_i)$ for $i = 1, 2$. Consider the DFA $A_1 \times A_2 = (Q, \Sigma, q_0, \delta, F)$ with

$$\begin{aligned}
Q &= Q_1 \times Q_2 \\
q_0 &= (q_{0,1}, q_{0,2}) \\
\delta((q, q'), a) &= (\delta_1(q, a), \delta_2(q', a)) \\
F &= F_1 \times F_2
\end{aligned}$$

We show by induction on x that

$$\delta^*((q, q'), x) = (\delta_1^*(q, x), \delta_2^*(q', x)) \quad \text{for all } x \in \Sigma^* \text{ and all } q \in Q_1, q' \in Q_2$$

base case: We have

$$\begin{aligned}\delta^*((q,q'),\epsilon) &= (q,q') \\ &= (\delta_1^*(q,\epsilon),\delta_2^*(q',\epsilon))\end{aligned}$$

step case: We assume that the property holds true for x, i.e.,

$$\delta^*((q,q'),x) = (\delta_1^*(q,x),\delta_2^*(q',x)) \quad \text{for all } q \in Q_1, q' \in Q_2$$

Now,

$$\begin{aligned}\delta^*((q,q'),xa) &= \delta(\delta^*((q,q'),x),a) \\ &= \delta((\delta_1^*(q,x),\delta_2^*(q',x)),a) \\ &= (\delta_1(\delta_1^*(q,x),a),\delta_2(\delta_2^*(q',x),a)) \\ &= (\delta_1^*(q,xa),\delta_2^*(q',xa))\end{aligned}$$

Hence, in particular, $\delta^*(q_0,x) = \delta^*((q_{0,1},q_{0,2}),x) = (\delta_1^*(q_{0,1},x),\delta_2^*(q_{0,2},x))$.

Thus,

$$\begin{aligned}L(A_1 \times A_2) &= \{w \in \Sigma^* \mid \delta^*(q_0,x) \in F\} \\ &= \{w \in \Sigma^* \mid (\delta_1^*(q_{0,1},x),\delta_2^*(q_{0,2},x)) \in F_1 \times F_2\} \\ &= \{w \in \Sigma^* \mid \delta_1^*(q_{0,1},x) \in F_1\} \cap \{w \in \Sigma^* \mid \delta_2^*(q_{0,2},x) \in F_2\} \\ &= L(A_1) \cap L(A_2) \\ &= L_1 \cap L_2\end{aligned}$$

Therefore, $L_1 \cap L_2$ is regular as well. □

Proposition 7.3.4
If L_1 and L_2 are regular languages, then so is $L_1 \cup L_2$.

Proof. From set theory we know that $L_1 \cup L_2 = \overline{\overline{L_1} \cap \overline{L_2}}$. Using the closure of regular languages under complementation and intersection, the claim follows.

Alternatively, we can explicitly construct an DFA for $L_1 \cap L_2$ from DFAs for L_1 and L_2. The idea is similar to the construction of the automaton in the proof of Proposition 7.3.3. Thus, let $A_i = (Q_i, \Sigma, q_{0,i}, \delta_i, F_i)$ be a DFA that accepts L_i, for $i = 1, 2$. Consider the DFA $A_1 \otimes A_2 = (Q, \Sigma, q_0, \delta, F)$ with

$$\begin{aligned}Q &= Q_1 \times Q_2 \\ q_0 &= (q_{0,1}, q_{0,2}) \\ \delta((q,q'),a) &= (\delta_1(q,a),\delta_2(q',a)) \\ F &= (F_1 \times Q_2) \cup (Q_1 \times F_2)\end{aligned}$$

Then,

$$\begin{aligned}
L(A_1 \otimes A_2) &= \{w \in \Sigma^* \mid \delta^*(q_0, x) \in F\} \\
&= \{w \in \Sigma^* \mid (\delta_1^*(q_{0,1}, x), \delta_2^*(q_{0,2}, x)) \in (F_1 \times Q_2) \cup (Q_1 \times F_2)\} \\
&= \{w \in \Sigma^* \mid \delta_1^*(q_{0,1}, x) \in F_1\} \cup \{w \in \Sigma^* \mid \delta_2^*(q_{0,2}, x) \in F_2\} \\
&= L(A_1) \cup L(A_2) \\
&= L_1 \cup L_2
\end{aligned}$$

Therefore, $L_1 \cup L_2$ is regular as well. □