2.1 Peano axioms (addendum)

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2.1 Peano axioms

Here, we give an alternative approach to constructing the natural numbers. Using this approach, Axioms 2.1.3, 2.1.6, 2.1.7, 2.1.8 and 2.1.9 can be proved, i.e., they are not taken as axioms.

Starting with the empty set $\emptyset$, we can define the first four natural numbers as

$$\begin{array}{lllllllll} 0 & & & & & & & = & \emptyset \\ 1 & = & 0 \cup \{0\} & & & & & = & \{0\} \\ 2 & = & 1 \cup \{1\} & = & \{0\} \cup \{\{0\}\} & = & \{0, \{0\}\} & = & \{0, 1\} \\ 3 & = & 2 \cup \{2\} & = & \{0, 1\} \cup \{\{0, 1\}\} & = & \{0, 1, \{0, 1\}\} & = & \{0, 1, 2\} \end{array}$$

Expressed just in terms of the empty set, the above are

$$\begin{array}{lll} 0 & = & \emptyset \\ 1 & = & \{\emptyset\} \\ 2 & = & \{\emptyset, \{\emptyset\}\} \\ 3 & = & \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} \end{array}$$

Clearly, the numbers just defined are all distinct. Other interesting and unusual consequences of the above definitions are $0 \in 1 \in 2 \in 3$ and $0 \subset 1 \subset 2 \subset 3$.

In the following, we show how to use this approach in order to define all natural numbers.

2.1.1 Transitive sets

In general, the notions of "element" and "subset" need to be carefully distinguished. For *transitive* sets, however, each element is also a subset.

Definition 2.1.12
Let A be a set. Then A is TRANSITIVE *if $x \in A$ implies $x \subseteq A$ for all x.*

Example 2.1.13

1. The set $A = \{\emptyset, \{\emptyset\}\}$ is transitive. Its elements are $\emptyset$ and $\{\emptyset\}$. Its subsets are $\emptyset$, $\{\emptyset\}$, $\{\{\emptyset\}\}$ and $\{\emptyset, \{\emptyset\}\}$. Hence, every element is also a subset.

2. The set $A = \{\emptyset, \{\{\emptyset\}\}\}$ is not transitive. Its elements are $\emptyset$ and $\{\{\emptyset\}\}$. Its subsets are $\emptyset$, $\{\emptyset\}$, $\{\{\{\emptyset\}\}\}$, and $\{\emptyset, \{\{\emptyset\}\}\}$. Thus, its element $\{\{\emptyset\}\}$ is not a subset.

3. The sets $0, 1, 2, 3$ defined above are all transitive.

2.1.2 The set of natural numbers

Next, we define the set ω of natural numbers. Central to this is the concept of an *inductive* set.

Definition 2.1.14
Let A be a set. Then A is INDUCTIVE *if*

1. $\emptyset \in A$,

2. $x \in A \implies x \cup \{x\} \in A$ *for all* x.

For any set x, denote its *successor* by $x^+ = x \cup \{x\}$. Then, an inductive set is supposed to contain the empty set and the successor of any of its members. We need an axiom that ensures the existence of an inductive set.

Axiom 2.1.15
There exists an inductive set.

We are interested in the "smallest" inductive set.

Definition 2.1.16
Let $\mathcal{I} = \{A \mid A \text{ is inductive}\}$*. Then* $\omega := \bigcap \mathcal{I} = \{x \mid x \in A \text{ for all } A \in \mathcal{I}\}$.

Proposition 2.1.17
1. $\omega \subseteq A$ *for all* $A \in \mathcal{I}$.

2. ω *is inductive.*

Proof.

1. Let $x \in \omega$. Then, by the definition of ω, we get $x \in A$ for all $A \in \mathcal{I}$. Thus, $\omega \subseteq A$ for all $A \in \mathcal{I}$.

2. Since all $A \in \mathcal{I}$ are inductive, all of them contain $\emptyset$. Thus, $\emptyset \in \omega$ as well.

   Let $x \in \omega$. Then $x \in A$ for all $A \in \mathcal{I}$. Since all A are inductive, $x \cup \{x\} \in A$ for all of them. Thus, $x \cup \{x\} \in \omega$ as well. □

Now, a *natural number* is defined to be any element of the set ω.

2.1.3 Peano axioms as propositions

We show that the Peano axioms are consequences of the definition of ω. For this, we identify 0 with the empty set. The successor of the natural number n is just $n^+ = n \cup \{n\}$.

Proposition 2.1.18 (cf. Axioms 2.1.3 and 2.1.6)

1. $\emptyset \in \omega$
2. $n^+ \in \omega$ *for all* $n \in \omega$

Proof. This just restates conditions 1 and 2 of an inductive set. □

Proposition 2.1.19 (cf. Axiom 2.1.8)
$n^+ \neq \emptyset$ *for all* $n \in \omega$.

Proof. For this, we notice that n^+ contains n since $n^+ = n \cup \{n\}$. In particular, n^+ is nonempty and thus distinct from $\emptyset$, which is empty. □

Proposition 2.1.20 (cf. Axiom 2.1.9)
Let $A \subseteq \omega$. *Then* $A = \omega$ *provided that*

base case: $\emptyset \in A$

step case: $n^+ \in A$ *for all* $n \in A$

Proof. Let A be any set satisfying the base and step cases. Since these cases correspond to conditions 1 and 2 in the definition of inductive sets, A is inductive. Hence, $\omega \subseteq A$.

Since $A \subseteq \omega$ by assumption, $A = \omega$ follows. □

Proving that ω satisfies Axiom 2.1.7 requires some preparation.

Proposition 2.1.21
Let $m, n \in \omega$ *such that* $m \in n^+$. *Then either* $m \in n$ *or* $m = n$, *but not both.*

Proof. Clearly, $m \in n^+$ implies $m \in n$ or $m = n$ by the definition of n^+. Thus, we need to show that not both of them can be true at the same time. For this, assume $m \in n$ and $m = n$. Then, $n \in n$. But this is impossible since no set can be a member of itself (this follows from the *Axiom of Foundation* in set theory). □

Proposition 2.1.22
Let $n \in \omega$. *Then* n *is transitive.*

Proof. We proof the proposition by induction on n.

base case: Since $\emptyset$ has no elements, $\emptyset$ is clearly transitive.

step case: Assume that n is transitive. We need to show that n^+ is transitive as well.

For every $m \in n^+$, either $m \in n$ or $m = n$. If $m \in n$, then $m \subseteq n$ since n is transitive. Thus, $m \subseteq n^+ = n \cup \{n\}$.

If $m = n$, then $m \subseteq n^+ = n \cup \{n\}$ as well. □

Proposition 2.1.23 (cf. Axiom 2.1.7)
$m^+ = n^+ \implies m = n$ *for all* $m, n \in \omega$.

Proof. Let $m, n \in \omega$ such that $m^+ = n^+$. Since $n \in n^+$ and $m^+ = n^+$, we have $n \in m^+$. By what we have shown above, either $n \in m$ or $n = m$, but not both. Similarly, since $m \in m^+$ and $m^+ = n^+$, we have either $m \in n$ or $m = n$, but not both.

If $m = n$, we are done. Otherwise, we have $m \in n$ and $n \in m$. Since both m and n are transitive, we get $m \subseteq n$ and $n \subseteq m$. Thus, $m = n$. □