1.4 Ordered pairs (addendum)

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1.4 Ordered pairs

Definition 1.4.5
Let a and b be objects. The ORDERED PAIR *with a as the first object and b as the second object, written $(a,b)_K$, is the set*

$$\{\{a\},\{a,b\}\}.$$

Remark 1.4.6
The subscript in $(a,b)_K$ is in honor of Kazimierz Kuratowski who first proposed this definition in 1921.

With this definition of ordered pairs, equality of ordered pairs reduces to equality of sets, hence Axiom 1.4.2 becomes a proposition.

Proposition 1.4.7
Let a,b,c,d be objects. Then the following are equivalent:

1. $(a,b)_K = (c,d)_K$
2. $a = c$ *and* $b = d$

Proof.

"2. $\implies$ 1.": Assume $a = c$ and $b = d$. Then $\{a\} = \{c\}$ and $\{a,b\} = \{c,d\}$. Thus, $\{\{a\},\{a,b\}\} = \{\{c\},\{c,d\}\}$, i.e, $(a,b)_K = (c,d)_K$.

"1. $\implies$ 2.": Assume $(a,b)_K = (c,d)_K$. We make a case distinction on whether $a = b$ or not.

$a = b$: If $a = b$, then $(a,b)_K = \{\{a\},\{a,a\}\} = \{\{a\},\{a\}\} = \{\{a\}\}$. Since $(a,b)_K = (c,d)_K$ we have $(c,d)_K = \{\{c\},\{c,d\}\} = \{\{a\}\}$. Thus, $\{c\} = \{a\}$ and $\{c,d\} = \{a\}$. Therefore, $c = d = a = b$.

$a \neq b$: Since $(a,b)_K = (c,d)_K$ we have $\{\{a\},\{a,b\}\} = \{\{c\},\{c,d\}\}$.
If $\{c\} = \{a,b\}$, then $c = a = b$, which contradicts $a \neq b$. Thus, $\{c\} = \{a\}$ and $c = a$.
If $\{c,d\} = \{a\}$, then $c = d = a$ and $(c,d)_K = \{\{c\},\{c,d\}\} = \{\{a\},\{a,a\}\} = \{\{a\},\{a\}\} = \{\{a\}\}$. Since $a \neq b$, we have $\{\{a\}\} \neq \{\{a\},\{a,b\}\}$, which contradicts $(a,b)_K = (c,d)_K$. Thus, $\{c,d\} = \{a,b\}$.
If $d = a$, then $\{c,d\} = \{c,a\} = \{a,a\} = \{a\}$, which we just showed to be impossible. Thus, $d = b$.
In summary, $a = c$ and $b = d$. ∎