Homework 1

Due: Sept 9, 2003

1. Simplify the following linear algebra expressions as far as possible (essentially, remove as many parenthases as you can). For each, describe the shape of each element as completely as you can (e.g., "if X is an $n \times n$ square matrix, then Y must be an $n \times m$ rectangular matrix, Z must be the same shape as X, P must be an m column vector, and Q can be an arbitrary rectangular matrix"). What is the shape of the resulting expression?

   (a) $(X + Y)^{\mathsf{T}}(X + Y)$

   (b) $((OR)^{-1}(N^{-1}F^{-1}))^{-1}D$

   (c) $((A + B + C)^{\mathsf{T}}D(E + F - G - H)^{\mathsf{T}}I(J + K))^{\mathsf{T}}$

   (d) $(X + Y)^{\mathsf{T}}(Q + Z)(Q + Z)^{\mathsf{T}}(X + Y)$ where all elements are column vectors.

   (e) $(X + Y)^{\mathsf{T}}(Q + Z)(Q + Z)^{\mathsf{T}}(X + Y)$ where all elements are arbitrary rectangular matrices.

2. Let $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{b}$ be column vectors of length n, A be an $n \times n$ square matrix, and c be a scalar constant. Consider the function defined by:

$$f(\mathbf{x}) = \mathbf{x}^{\mathsf{T}}A\mathbf{x} + \mathbf{x}^{\mathsf{T}}\mathbf{y}\mathbf{y}^{\mathsf{T}}\mathbf{x} + \mathbf{x}^{\mathsf{T}}\mathbf{b} + c$$

   (a) What is the shape of $f(\mathbf{x})$? (I.e., the shape of the result of the function.)

   (b) How many stationary points does this function have? Find an expression that describes the stationary point(s).

   (c) Suppose I tell you that a particular stationary point of this function is a maximum. What can you infer about the constituents of the function? What if it's a minimum?

3. Consider the following system of equations

$$V_i = R_i + \gamma \sum_{j=1}^{n} T_{ij}V_j$$

   where i and j both range $1..n$ and all quantities (V_i, R_i, etc.) are scalar.

   (a) Write this system in linear algebraic notation (i.e., in terms of vectors and matrices rather than scalars). What "shape" is each – i.e., list which are vectors, which matrices, which scalars.

   (b) Using your answer for 3a, solve for V in terms of the other quantities.

4. Given the following distribution over three discrete random variables X, Y, and Z, each taking on the values 0 or 1:

$X = 0$

	$Z = 0$	$Z = 1$
$Y = 0$	0	$\frac{1}{6}$
$Y = 1$	$\frac{1}{15}$	$\frac{1}{10}$

$X = 1$

	$Z = 0$	$Z = 1$
$Y = 0$	0	$\frac{1}{3}$
$Y = 1$	$\frac{2}{15}$	$\frac{1}{5}$

Find:

(a) The joint distributions of X and Y, X and Z, and Y and Z.

(b) The marginal distributions of X, Y, and Z.

(c) The conditional distribution of X given Y, X given Z, and Y given Z.

(d) For each pair of RVs in this set, explain whether those two variables are statistically independent (including why).

5. The joint distribution of a simple two-dimensional Gaussian random variable is given by the PDF:

$$f_{XY}(x, y) = \frac{1}{2\pi\sqrt{5}} e^{-\frac{1}{2}\left(\frac{1}{5}x^2 + y^2\right)}$$

(a) Describe the general structure of this PDF (e.g., what shape are the isopotential surfaces in 2-d space?)

(b) Find $f_X(x)$ and $f_Y(y)$ (i.e., the marginal distributions of x and y). Describe each random variable.

(c) Are x and y statistically independent? Why or why not?

(d) Find $f_{X|Y}(x|y = 3)$ and $f_{Y|X}(y|x = -1)$ (i.e., the conditional distributions of x given a fixed value of y and vice-versa).

6. The joint distribution of a more complicated two-dimensional Gaussian random variable is given by the PDF:

$$f_{XY}(x, y) = \frac{1}{2\pi\sqrt{5}} e^{-\frac{1}{2}\left(\frac{2}{5}x^2 + \frac{4}{5}y^2 + \frac{6}{5\sqrt{3}}xy + \left(\frac{-4\sqrt{3}+12}{5\sqrt{3}}\right)x + \left(\frac{16\sqrt{3}-6}{5\sqrt{3}}\right)y + \frac{18\sqrt{3}-12}{5\sqrt{3}}\right)}$$

(a) Find the marginal distributions of x and y and describe each random variable.

(b) Are these variables statistically independent? Why or why not?