For Distinguishing Conjugate Hidden Subgroups, the Pretty Good Measurement is as Good as it Gets

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Abstract

Recently Bacon, Childs and van Dam showed that the “pretty good measurement” (PGM) is optimal for the Hidden Subgroup Problem on the dihedral group D_n in the case where the hidden subgroup is chosen uniformly from the n involutions. We show that, for any group and any subgroup H , the PGM is the optimal one-register experiment in the case where the hidden subgroup is a uniformly random conjugate of H . We go on to show that when H forms a Gel’fand pair with its parent group, the PGM is the optimal measurement for any number of registers. In both cases we bound the probability that the optimal measurement succeeds. This generalizes the case of the dihedral group, and includes a number of other examples of interest.

1 The Hidden Conjugate Problem

Consider the following special case of the Hidden Subgroup Problem, called the *Hidden Conjugate Problem* in [16]. Let G be a group, and H a non-normal subgroup of G ; denote conjugates of H as $H^g = g^{-1}Hg$. Then we are promised that the hidden subgroup is H^g for some g , and our goal is to find out which one.

The usual approach is to prepare a uniform superposition over the group, entangle the group element with a second register by calculating or querying the oracle function, and then measure the oracle function. This yields a uniform superposition over a random left coset of the hidden subgroup,

$$|cH^g\rangle = \frac{1}{\sqrt{|H|}} \sum_{h \in H^g} |ch\rangle .$$

Rather than viewing this as a pure state where c is random, we may treat this as a classical mixture over left cosets, giving the mixed state with density matrix

$$\rho_g = \frac{1}{|G|} \sum_{c \in G} |cH^g\rangle \langle cH^g| . \quad (1.1)$$

We then wish to find a positive operator-valued measurement (POVM) to identify g .

A POVM consists of a set of positive *measurement operators* $\{E_i\}$ that obey the completeness condition

$$\sum_i E_i = \mathbb{1} . \quad (1.2)$$

If we are trying to distinguish ℓ density matrices $\{\rho_i \mid 1 \leq i \leq \ell\}$, and if ρ_i is chosen with probability p_i , the probability that the POVM gives the right answer is

$$P_{\text{success}} = \sum_i p_i \text{tr } E_i \rho_i . \quad (1.3)$$

A theorem of Yuen, Kennedy and Max [22] and Holevo [9] states that P_{success} is maximized if and only if the following conditions hold for every i :

$$\left(\sum_j p_j \rho_j E_j - p_i \rho_i \right) E_i = 0 , \quad (1.4)$$

$$\sum_j p_j \rho_j E_j = \sum_j p_j E_j \rho_j , \text{ and} \quad (1.5)$$

$$\sum_j p_j \rho_j E_j \geq p_i \rho_i , \quad (1.6)$$

where we write $A \geq B$ if $A - B$ is positive semidefinite. These conditions come from recognizing that maximizing P_{success} subject to the completeness condition (1.2) gives a semidefinite program. Ip [11] used this fact to show that Shor's algorithm is optimal for the hidden subgroup problem on the cyclic group $\mathbb{Z}_n$.

In a beautiful recent paper, Bacon, Childs and van Dam [1] consider the hidden conjugate problem for the dihedral group D_n where H is an order-2 subgroup; that is, when the hidden subgroup consists of the identity and one of the n involutions. They consider entangled measurements over multiple registers each of which contains a coset state, and show that the so-called “pretty good measurement” [7], defined below, is in fact optimal for any number of registers. They then show that this optimal measurement is related to random cases of the Subset Sum problem (see also Regev [18]). Other prior work includes Eldar, Megretski, and Verghese [4], who showed that the PGM is optimal for certain families of density matrices related by group symmetries; see references in that paper for some other cases for which the PGM is optimal.

In this paper we point out that for any subgroup H of a group G , the pretty good measurement is the optimal one-register experiment for finding a hidden conjugate of H when all conjugates are equally likely. We then write a general expression for the probability P_{success} that this measurement identifies the hidden conjugate in a single experiment on a coset state. We then go on to show that when G and H form a Gel'fand pair, the pretty good measurement is in fact the optimal measurement on *any number of registers*. This recovers the optimality result of [1] as a special case, and establishes optimal measurements for a variety of other interesting group-subgroup pairs, including the subgroups of the affine groups studied in [16].

We use the machinery of representation theory; we refer the reader to [6, 20] or to the review in our paper [14] for an introduction and for notation.

2 The one-register case

2.1 The pretty good measurement is optimal

The *pretty good measurement* (PGM), also known as the *least squares measurement*, is defined as follows [7]. Given a set of density matrices ρ_i with associated probabilities p_i , let

$$M = \sum_i p_i \rho_i.$$

Then the PGM associated with this family of mixed states is $\{E_i\}$, where the measurement operator E_i is defined as

$$E_i = p_i M^{-1/2} \rho_i M^{-1/2} \quad (2.1)$$

where the inverse $M^{-1/2}$ is defined on the image of M ; that is, $M^{-1/2}$ is the unique positive operator such that $(M^{-1/2})^2 M$ is the projection operator Π onto the image of M . If M has full rank, it is easy to see that this choice of $\{E_i\}$ satisfies the completeness condition (1.2); if the image of M is a proper subspace, we satisfy the completeness condition by adding an additional measurement which projects onto its orthogonal complement.

We will show that the optimality conditions (1.4), (1.5), and (1.6) hold for the PGM for the family of density matrices $\{\rho_g\}$ defined above when endowed with the uniform distribution $p_g = 1/|G|$. First we derive the structure of the PGM. Observe that the mixed state $\rho_g = |G|^{-1} \sum_c |cH^g\rangle \langle cH^g|$ has the property that

$$\rho_g^2 = \frac{|H|}{|G|} \rho_g \quad (2.2)$$

so that ρ_g is a projection operator scaled by the constant $|H|/|G|$ (this follows from the fact that the uniform distribution on any subgroup is its own square under convolution). As $\text{tr} \rho_g = 1$, we must have $\text{rk} \rho_g = |G|/|H|$, the index of H in G .

Note that ρ_g commutes with the left action of G , since it is “symmetrized” over all left cosets. By Schur’s lemma, ρ_g is block diagonal, with blocks corresponding to the decomposition of $\mathbb{C}[G]$ into bi-invariant spaces. Furthermore, the block corresponding to the irreducible representation σ has form $A_\sigma \otimes \mathbb{1}_{d_\sigma}$, where $\mathbb{1}_{d_\sigma}$ acts within each left-invariant space. Recalling (2.2), A_σ is a rescaled projection operator, and we may write $\rho_g = \bigoplus_{\sigma \in \widehat{G}} \rho_g^\sigma$ where

$$\rho_g^\sigma = \frac{|H|}{|G|} \pi_g^\sigma \otimes \mathbb{1}_{d_\sigma} ; \quad (2.3)$$

here π_g^σ is the projection operator

$$\pi_g^\sigma = \frac{1}{|H|} \sum_{h \in H^g} \sigma(h) . \quad (2.4)$$

Since $\text{rk} \pi_g^\sigma$ does not depend on g , we denote it simply as $\text{rk} \pi^\sigma$. Then the following lemma describes the structure of the PGM for $\{\rho_g\}$. Since $\rho_g = \rho_h$ whenever g and h are in the same coset of the normalizer $\text{Norm}(H) = \{g \mid H^g = H\}$, without loss of generality we assume that the POVM gives a uniformly random element of some coset of $\text{Norm}(H)$, and we count it as having succeeded if it gives an element of the correct coset. Thus we will multiply P_{success} by the index of $\text{Norm}(H)$ below.

Lemma 1. *For the family $\{\rho_g\}$ of density matrices corresponding to random left cosets of conjugate subgroups H^g with the uniform distribution on g , the pretty good measurement operators E_g are given by*

$$E_g = \bigoplus_{\sigma \in \widehat{G}} E_g^\sigma$$

and, for each $\sigma \in \widehat{G}$,

$$E_g^\sigma = \frac{d_\sigma}{|G| \text{rk} \pi^\sigma} \pi_g^\sigma \otimes \mathbb{1}_{d_\sigma} \quad (2.5)$$

where π_g^σ is defined as in (2.3).

Proof. For each $x \in G$, let L_x and R_x denote the unitary operators that carry out left and right group multiplication by x . Note that left cosets are mapped to each other by L_x and, in particular, $|cH\rangle = L_c |H\rangle$. Note furthermore that R_x maps left cosets of one conjugate onto left cosets of another conjugate, e.g., $|H^g\rangle = R_g |g^{-1}H\rangle$, and that L_x commutes with R_y for all x, y .

We have $p_g = 1/|G|$ for all g . Now if we write

$$\begin{aligned} M &= \frac{1}{|G|} \sum_g \rho_g = \frac{1}{|G|^2} \sum_{c,g} |cH^g\rangle \langle cH^g| = \frac{1}{|G|^2} \sum_{c,g} |cHg\rangle \langle cHg| \\ &= \frac{1}{|G|^2} \sum_{c,g} L_c R_g |H\rangle \langle H| R_g^\dagger L_c^\dagger \end{aligned}$$

(where in the third equality we replace cg^{-1} with c), we see that M commutes with L_x and R_x for all $x \in G$. That is, summing over both the left coset and the choice of conjugate “symmetrizes” M on both the left and the right. It follows by Schur’s lemma that M takes the form

$$M = \bigoplus_{\sigma \in \widehat{G}} M^\sigma$$

where M^σ is a scalar multiple of the identity operator for each σ . As

$$M_\sigma = \frac{1}{|G|} \sum_g \rho_g^\sigma = \frac{|H|}{|G|^2} \sum_g \pi_g^\sigma \otimes \mathbb{1}_{d_\sigma} ,$$

by taking traces we conclude that

$$M^\sigma = \left(\frac{|H|}{|G|} \frac{\mathbf{rk} \pi^\sigma}{d_\sigma} \right) \mathbb{1}_{d_\sigma^2} .$$

Similarly, ρ_g is block-diagonal, as it commutes with L_x (though not with R_x unless H is normal). Therefore, M commutes with ρ_g for each g , and (2.1) becomes

$$E_g = \frac{1}{|G|} M^{-1} \rho_g \quad (2.6)$$

giving, in each irreducible block,

$$E_g^\sigma = \frac{d_\sigma}{|H| \mathbf{rk} \pi^\sigma} \rho_g^\sigma = \frac{d_\sigma}{|G| \mathbf{rk} \pi^\sigma} \pi_g^\sigma \otimes \mathbb{1}_{d_\sigma} \quad (2.7)$$

which completes the proof. $\square$

We now give our proof that the PGM is optimal for the hidden conjugate problem for any G and H . This follows simply from the fact that M , and therefore E_g , commutes with ρ_g for each g .

Theorem 2. *For the family $\{\rho_g\}$ of density matrices corresponding to random left cosets of conjugate subgroups H^g with the uniform distribution on g , the pretty good measurement $\{E_g\}$ optimizes the probability of correctly measuring g .*

Proof. Since the E_g and ρ_g are block-diagonal (according to the same decomposition of $\mathbb{C}[G]$), it suffices to confirm the optimality criteria (1.4), (1.5), and (1.6) in each block, i.e., for each $\sigma \in \hat{G}$.

First, (1.5) holds trivially since E_g^σ is proportional to ρ_g^σ for each g , and so commutes with it. As for condition (1.6), observe that from (2.2) and (2.7) we have

$$\rho_g^\sigma E_g^\sigma = \frac{d_\sigma}{|G| \mathbf{rk} \pi^\sigma} \rho_g^\sigma \quad (2.8)$$

and so for any $h \in G$, recalling that $p_g = 1/|G|$ for all g , we have

$$\frac{1}{|G|} \sum_g \rho_g^\sigma E_g^\sigma = \frac{d_\sigma}{|G| \mathbf{rk} \pi^\sigma} M^\sigma = \frac{|H|}{|G|^2} (\mathbb{1}_{d_\sigma} \otimes \mathbb{1}_{d_\sigma}) \geq \frac{|H|}{|G|^2} (\pi_h^\sigma \otimes \mathbb{1}_{d_\sigma}) = \frac{1}{|G|} \rho_h$$

since $\mathbb{1}_{d_\sigma} \geq \pi_h^\sigma$. Finally, for any $h \in G$

$$\left(\sum_g \rho_g^\sigma E_g^\sigma - \rho_h^\sigma \right) E_h^\sigma = \frac{|H|}{|G|} [(\mathbb{1}_{d_\sigma} - \pi_h^\sigma) \otimes \mathbb{1}_{d_\sigma}] E_h^\sigma = \frac{|H| d_\sigma}{|G|^2 \mathbf{rk} \pi^\sigma} [(\mathbb{1}_{d_\sigma} - \pi_h^\sigma) \pi_h^\sigma] \otimes \mathbb{1}_{d_\sigma} = 0$$

since $(\mathbb{1} - \pi)\pi = 0$ for any projection operator π . Again recalling $p_g = 1/|G|$ establishes (1.4) and completes the proof. $\square$

Note that Lemma 1 and Theorem 2 imply that, as pointed out before [11, 13, 14], the optimal measurement consists of first measuring the representation name σ , and then performing an additional measurement M_g^σ inside the bi-invariant space corresponding to σ .

2.2 Partial measurements

Suppose that rather than trying to identify the conjugate exactly, we wish to learn some partial information about it. To learn one bit, for instance, we would divide the set of conjugates into two equal subsets, and combine the ρ_g into two mixed states ρ_0 and ρ_1 consisting of mixtures of those in the two subsets. The next theorem shows that the PGM is optimal for any such partial measurement as long as each subset of the set of conjugates has probability proportional to its size.

Theorem 3. *Let $C(H) = \{H^g\}$ be partitioned into disjoint sets C_i , $1 \leq i \leq \ell$. Let*

$$\rho_i = \frac{1}{|C_i|} \sum_{g: H^g \in C_i} \rho_g$$

where ρ_g is as in (1.1), and let $p_i = |C_i|/|C(H)|$. Then the pretty good measurement for the family $\{\rho_i\}$ with probabilities p_i is the family of operators $\{E_i\}$

$$E_i = \sum_{g: H^g \in C_i} E_g ,$$

where E_g is given by Lemma 1; this measurement is optimal.

Proof. The proof is exactly the same as that of Lemma 1 and Theorem 2, except that for each i we sum over the g with $H^g \in C_i$. $\square$

2.3 The probability of success

As a corollary to Theorem 2, we can determine the optimal success probability. Let $C(H)$ denote the set of conjugates of H . For a group G the *Plancherel measure* is the probability distribution on $\widehat{G}$ assigning $\sigma \in \widehat{G}$ the probability $d_\sigma^2/|G|$; this is the fraction, dimensionwise, of $\mathbb{C}[G]$ consisting of the bi-invariant subspace corresponding to σ . For a set $S \subset \widehat{G}$, we let $\text{Planch}(S)$ denote the probability of observing an element of S according to the Plancherel measure. We remark that the Plancherel measure is precisely the probability distribution obtained by performing weak Fourier sampling when the hidden subgroup is trivial, since in that case the state ρ is completely mixed.

Given a subgroup H of G , let $S_H \subseteq \widehat{G}$ denote the set of irreducible representations for which the projection operator $\pi_H^\sigma = |H|^{-1} \sum_{h \in H} \sigma(h)$ is nonzero, and let $H_G = \cap_g H^g$ denote the largest normal subgroup contained in H . Then the following theorem gives the success probability of the optimal one-register experiment.

Theorem 4. *Given a group G and a subgroup H , the probability that the optimal single-register measurement correctly identifies a uniformly random conjugate of H is*

$$P_{\text{success}} = \frac{|H|}{|C(H)|} \text{Planch}(S_H) \leq \frac{|H|}{|C(H)| \cdot |H_G|} . \quad (2.9)$$

Proof. Let us say that $g \sim g'$ if $H^g = H^{g'}$ (and so $\rho_g = \rho_{g'}$). Then P_{success} is the expectation

$$P_{\text{success}} = \text{Exp}_g \sum_{g' \sim g} \text{tr } E_{g'} \rho_g ,$$

where g is selected uniformly in G . Since $g \sim g'$ if and only if g and g' are in the same (right) coset of the normalizer $\text{Norm}(H) = \{g \mid H^g = H\}$, we have

$$P_{\text{success}} = |\text{Norm}(H)| \cdot \text{Exp}_g \text{tr } E_g \rho_g = |\text{Norm}(H)| \cdot \text{Exp}_g \sum_{\sigma} \text{tr } E_g^\sigma \rho_g^\sigma .$$

Considering (2.3) and (2.8), we have

$$\text{tr } E_g^\sigma \rho_g^\sigma = \frac{|H| d_\sigma^2}{|G|^2} = \frac{|H|}{|G|} \text{Planch}(\sigma) ,$$

but only for those σ where ρ_g^σ and E_g^σ are nonzero, i.e., those for which $\mathbf{rk} \pi^\sigma > 0$. Thus, we conclude that

$$P_{\text{success}} = \frac{|\text{Norm}(H)||H|}{|G|} \cdot \sum_{\sigma \in S_H} \text{Planch}(\sigma) = \frac{|H|}{|C(H)|} \cdot \text{Planch}(S_H)$$

where we recall that $|C(H)| = |G|/|\text{Norm}(H)|$.

Now, note that for any subgroup $K \subseteq H$, we have $S_H \subseteq S_K$ since any σ that annihilates K also annihilates H . In addition, if K is normal, recall that for any σ we have either $\pi_K^\sigma = 0$ or $\pi_K^\sigma = \mathbb{1}$. It follows that $\sum_{\sigma \in S_K} d_\sigma^2 = \sum_{\sigma \in S_K} d_\sigma \mathbf{rk} \pi_K^\sigma = \mathbf{rk} \pi_K^R$ where R is the regular representation, and since $\mathbf{rk} \pi_K^R = |G|/|K|$ we have $\text{Planch}(S_K) = 1/|K|$. Thus $\text{Planch}(S_H) \leq 1/|H_G|$, completing the proof of (2.9). $\square$

Note that if we observe any $\sigma \notin S_H$, we know that the promise that the hidden subgroup is a conjugate of H has been violated. Thus, as in [1], if we are promised that the hidden subgroup is either trivial or a conjugate of H , we can complete the PGM with an additional measurement operator M_0 that projects onto the orthogonal complement of S_H , and conclude that the hidden subgroup is trivial if we observe the outcome M_0 .

It is interesting to compare Theorems 2 and 4 with known results on the hidden subgroup problem. For the dihedral group D_n where H is an order-2 subgroup, there are n conjugates, and S_H consists of all of $\widehat{G}$ except for the sign representation. Thus we have

$$P_{\text{success}} = \frac{2}{n} \left(1 - \frac{1}{2n} \right) .$$

On the other hand, for the affine group A_p , the maximal subgroup $H = \mathbb{Z}_p^*$ has p conjugates, and S_H includes all but the $p-2$ nontrivial one-dimensional representations. This gives

$$P_{\text{success}} = \frac{p-1}{p} \left(1 - \frac{p-2}{p(p-1)} \right) = 1 - \frac{2(p-1)}{p^2} .$$

Indeed, Moore, Rockmore, Russell and Schulman [16] gave an explicit algorithm using a von Neumann measurement that succeeds with constant probability. This algorithm can easily be modified to carry out the optimal POVM in polynomial time; see also Bacon, Childs and van Dam [2].

Now let us consider the case of the hidden subgroup problem relevant to Graph Isomorphism in the case of two rigid, connected graphs of size $n/2$. Here $G = S_n$ and H is the order-2 subgroup consisting of $n/2$ disjoint transpositions,

$$H = \{1, (12)(34) \cdots (n-1)n\}$$

of which there are $(n-1)!!$ conjugates, one for each perfect matching of n items. Using lemmas proved in [14], it is easy to show that for almost all representations σ (with respect to the Plancherel distribution) we have $\mathbf{rk} \pi^\sigma = (1 \pm o(1))d_\sigma/2$, so $\text{Planch}(S_H) = 1 - o(1)$. Thus we have

$$P_{\text{success}} = \frac{2}{(n-1)!!} (1 - o(1)) = n^{-n/2} e^{O(n)}$$

This can be generalized to other conjugacy classes using general character bounds due to Roichman [19]; see also Kempe and Shalev [12].

However, it should be emphasized that the fact that P_{success} is exponentially small does not mean that we need an exponential number of single-register experiments to solve the hidden conjugate problem. In particular, Ettinger and Høyer [5] showed that a polynomial number (i.e., $O(\log |G|) = O(\log n)$) of single-register experiments is enough to determine, information-theoretically, an involution in D_n . Thus our results here do not subsume the results of Moore, Russell and Schulman [14] and Moore and Russell [15], who showed that it takes an exponential number of single-register experiments, or a super-polynomial number of two-register experiments, to obtain *even a single bit of information* about the conjugate of H in S_n .

3 Multiregister measurements and Gel'fand pairs

For the multiregister experiment, we view states as elements of the Hilbert space $\mathbb{C}[G^k] = \mathbb{C}[G]^{\otimes k}$. We now have a random left coset of the subgroup $H^k \subset G^k$, and the corresponding mixed state is

$$\rho_g = \rho_g^{\otimes k} = \frac{1}{|G|^k} \sum_{\mathbf{c} \in G^k} |\mathbf{c}(H^g)^k\rangle \langle \mathbf{c}(H^g)^k| .$$

Since $\rho_g^{\otimes k}$ is symmetrized over left cosets, it commutes with left multiplication in G^k . Thus by Schur's lemma it is block-diagonal, where each block corresponds to a representation $\sigma = \sigma_1 \otimes \cdots \otimes \sigma_k$ of G^k , and each σ_i is an irreducible representation of G . Indeed, in a given such block we can write

$$\rho_g^\sigma = \rho_g^{\sigma_1} \otimes \cdots \otimes \rho_g^{\sigma_k} . \quad (3.1)$$

The situation in the multiregister case is complicated by the fact that, unlike the one-register case, M does not generally commute with ρ_g . Indeed, they do not commute even for the two-register case in the dihedral group. We note in passing that they do commute in a few special cases: for instance, when H is generated by an involution that commutes with its conjugates. However, this is not a very interesting case, since then H and its conjugates generate an Abelian subgroup $K \subset G$, and we can distinguish them by solving the hidden subgroup problem on K .

However, we can still prove that the PGM is optimal in the case that G and H form a Gel'fand pair; we review the definition here, and also refer the reader to [21] for an introduction. Given a group G and a subgroup H , let $\mathcal{B} = \mathcal{B}_H(G)$ denote the collection of functions $f : G \rightarrow \mathbb{C}$ that are invariant under both left and right multiplication by H , i.e., such that $f(hg) = f(g) = f(gh)$ for all $g \in G$ and $h \in H$. This collection of *bi-invariant* functions forms a natural algebra under convolution, and $\mathcal{B}$ can be identified with the subalgebra of $\mathbb{C}[G]$ generated by elements corresponding to double cosets, $HgH = (\sum_{h \in H} h) \cdot g \cdot (\sum_{h \in H} h)$. Then the following criteria are equivalent, and the pair (G, H) is said to be *Gel'fand* if any of them hold:

1. $\mathcal{B}$ is commutative.
2. The induced representation $\text{Ind}_H^G \mathbb{1}$ contains no more than one copy of any particular $\sigma \in \widehat{G}$.
3. For any $\sigma \in \widehat{G}$ and any $f \in \mathcal{B}$, the Fourier transform $\hat{f}(\sigma) = \sum_g f(g)\sigma(g)$ has rank at most one.

The third criterion is the one most relevant to our analysis. Suppose (G, H) is a Gel'fand pair; then since the uniform distribution on H is an element of $\mathcal{B}$, for any $\sigma \in \widehat{G}$ the projection operator $\pi^\sigma = |H|^{-1} \sum_{h \in H} \sigma(h)$ has rank at most one. Since this is also true of its conjugates $\pi_g^\sigma = \sigma(g)^{-1} \pi^\sigma \sigma(g)$, we see that (G, H^g) is Gel'fand for all $g \in G$.

As stated above, Bacon, Childs and van Dam [1] showed that the pretty good measurement is optimal for the dihedral groups D_n when the hidden subgroup H is of order 2. Indeed, (D_n, H) is Gel'fand for these subgroups, and we generalize their result as follows.

Theorem 5. *For any number of registers $k > 0$, given the family $\{\rho_g\}$ of density matrices corresponding to random left cosets of conjugate subgroups $(H^g)^k \subset G^k$ with the uniform distribution on g , the pretty good measurement $\{E_g\}$ optimizes the probability of correctly measuring g .*

Proof. As before, we will show that the optimality conditions hold in each irreducible block, since the ρ_g , and therefore M and the E_g , are block-diagonal. Since the tensor product of rank-one operators has rank one, given $\sigma = \sigma_1 \otimes \cdots \otimes \sigma_k$ with $\sigma_i \in \widehat{G}$ for all i , from (3.1) we have either $\rho_g^\sigma = 0$ or

$$\rho_g^\sigma = |v_g\rangle \langle v_g| \otimes \mathbb{1}_{d_\sigma}$$

for some vector $v_g \in \sigma$. In the latter case we have $M^\sigma = m \otimes \mathbb{1}_{d_\sigma}$ where

$$m = \frac{1}{|G|} \left(\sum_g |v_g\rangle \langle v_g| \right)$$

and

$$\rho_g^\sigma E_g^\sigma = \frac{1}{|G|} \left(|v_g\rangle \langle v_g| m^{-1/2} |v_g\rangle \langle v_g| m^{-1/2} \right) \otimes \mathbb{1}_{d_\sigma} = \frac{C}{|G|} \rho_g^\sigma (M^\sigma)^{-1/2}$$

where C is defined as the inner product

$$C = \langle v_g | m^{-1/2} |v_g\rangle \quad .$$

Similarly, $E_g^\sigma \rho_g^\sigma = \frac{C}{|G|} (M^\sigma)^{-1/2} \rho_g^\sigma$.

It is easy to see that C does not depend on g : if R_g denotes the unitary operator corresponding to right multiplication by the diagonal element $(g, \dots, g)$ in $\mathbb{C}[G^k]$, then $|v_g\rangle = R_g |v_1\rangle$ and m commutes with R_g . Thus $C = \langle v_g | m^{-1/2} |v_g\rangle = \langle v_1 | R_g m^{-1/2} R_g^\dagger |v_1\rangle = \langle v_1 | m^{-1/2} |v_1\rangle$. Then we have

$$\sum_g \rho_g^\sigma E_g^\sigma = \frac{C}{|G|} \sum_g \rho_g^\sigma (M^\sigma)^{-1/2} = C (M^\sigma)^{1/2} = \frac{C}{|G|} \sum_g (M^\sigma)^{-1/2} \rho_g^\sigma = \sum_g E_g^\sigma \rho_g^\sigma$$

confirming (1.5).

As for condition (1.6), recalling the equality $\sum_g \rho_g^\sigma E_g^\sigma = C (M^\sigma)^{1/2}$ just above, (1.6) is equivalent to the condition $C (M^\sigma)^{1/2} \geq \rho_g^\sigma$, and hence to $C m^{1/2} \geq |v_g\rangle \langle v_g|$. Furthermore, this holds if and only if

$$\langle v_g | C m^{1/2} |v_g\rangle \geq \langle v_g | v_g\rangle \langle v_g | v_g\rangle = \|v_g\|^4$$

or, expanding the definition of C ,

$$\langle v_g | m^{1/2} |v_g\rangle \langle v_g | m^{-1/2} |v_g\rangle \geq \|v_g\|^4 \quad . \quad (3.2)$$

We remark that if a positive semidefinite operator A is a linear combination $A = \sum_\alpha a_\alpha B_\alpha$ of positive semidefinite operators B_α with $a_\alpha \in \mathbb{R}^+$, then the kernel of A is $\bigcap_\alpha \ker B_\alpha$. As m is such a linear combination of operators $|v_g\rangle \langle v_g|$, it follows that $|v_g\rangle$ is orthogonal to $\ker m$, and therefore lies in the image of m . Thus we can regard $m^{1/2}$ and $m^{-1/2}$ as inverses, and the inequality (3.2) follows from the following claim:

Claim 6. *Let A be a positive operator on a finite dimensional Hilbert space V and let $|v\rangle \in V$. Then*

$$\langle v | A |v\rangle \langle v | A^{-1} |v\rangle \geq \|v\|^4 \quad .$$

Proof. For a vector $w = (w_1, \dots, w_n)$ of non-negative weights and a real number r , define

$$\mathcal{M}_r^w(x_1, \dots, x_n) = \left(\frac{\sum_i w_i x_i^r}{\sum_i w_i} \right)^{1/r}$$

to be the *weighted r -mean* of the positive vector $x = (x_1, \dots, x_n)$. The *power mean inequality* (cf. [8, §2.9]) asserts that for $r < s$ we have $\mathcal{M}_r^w(x) \leq \mathcal{M}_s^w(x)$.

The claim follows immediately from the from power mean inequality with $r = -1$ and $s = 1$. Specifically, let $B = \{|b_i\rangle\rangle$ be a spectral resolution of A , so that B is an orthogonal basis of eigenvectors for V , and let $\lambda_i > 0$ be the associated eigenvalues, so that $\lambda_i |b_i\rangle = A |b_i\rangle$. Writing $|v\rangle = \sum_i v_i |b_i\rangle$, we have

$$\langle v | A |v\rangle = \sum_i |v_i|^2 \lambda_i \quad \text{and} \quad \langle v | A^{-1} |v\rangle = \sum_i |v_i|^2 \lambda_i^{-1} \quad .$$

If we adopt the weights $w_i = |v_i|^2$, then $\sum_i w_i = \|v\|^2$ and the claim is equivalent to the the power mean inequality $\mathcal{M}_{-1}^w(\lambda_1, \dots, \lambda_n) \leq \mathcal{M}_1^w(\lambda_1, \dots, \lambda_n)$. $\square$

Returning to the proof of Theorem 5, it remains to establish (1.4). Consider

$$\begin{aligned} \left(\sum_g \rho_g^\sigma E_g^\sigma - \rho_h^\sigma \right) E_h^\sigma &= \frac{1}{|G|} \left[\left(C m^{1/2} - |v_h\rangle \langle v_h| \right) m^{-1/2} |v_h\rangle \langle v_h| m^{-1/2} \right] \otimes \mathbb{1}_{d_\sigma} \\ &= \frac{C}{|G|} \left[(\Pi - \mathbb{1}) |v_h\rangle \langle v_h| m^{-1/2} \right] \otimes \mathbb{1}_{d_\sigma} \end{aligned} \quad (3.3)$$

where Π is the projection operator onto the image of m . Since $|v_h\rangle$ lies in this image as discussed above, we have $\Pi |v_h\rangle = |v_h\rangle$ and (3.3) is identically zero. $\square$

As in the one-register case, we can distinguish the trivial subgroup from the conjugates of H by completing the PGM with a measurement M_0 that projects onto the complement of the image of M .

3.1 The Probability of Success

In this section we give an upper bound on the success probability of the optimal multiregister experiment for Gel'fand pairs. To prepare for this, we record a version of Holevo's theorem on the capacity of a quantum channel [10].

Lemma 7. *Let $R = \{\rho_i \mid i \in I\}$ be a family of density matrices treated as linear operators on the Hilbert space V . Let $E = \{E_i\}$ be a family of measurement operators on V for which $\sum_i E_i = \mathbb{1}$ and, for each $i \in I$, E_i is a scalar multiple of a projection operator of rank r . Then*

$$\text{Exp}_i \text{tr} E_i \rho_i \leq \frac{\dim H}{r|I|} ,$$

where the index i is chosen uniformly at random in I .

Proof. By assumption, we may write $E_i = \alpha_i \Pi_i$ where Π_i is a projection operator of rank r and $\alpha_i \geq 0$. As ρ_i is a density matrix, we may write $\rho_i = \sum_j p_j |v_j\rangle \langle v_j|$ where each $|v_j\rangle$ has unit length, each $p_j \in [0, 1]$, and $\sum_j p_j = 1$. Observe that

$$\text{tr} E_i \rho_i = \sum_j p_j \langle v_j | E_i | v_j \rangle \leq \sum_j p_j \|E_i\| = \|E_i\| = \alpha_i ,$$

where $\|A\|$ is the operator norm of A , given by $\|A\| = \max_{\mathbf{v} \neq 0} \|A\mathbf{v}\|/\|\mathbf{v}\|$. Hence

$$\text{Exp}_i \text{tr} E_i \rho_i \leq \frac{\sum_i \alpha_i}{|I|} .$$

Observe now that $\sum_i E_i = \mathbb{1}$ and hence that $\dim V = \text{tr} \mathbb{1} = \text{tr} \sum_i E_i = \sum_i \alpha_i r$; evidently $\sum_i \alpha_i = \dim V/r$, which completes the proof. $\square$

Theorem 8. *Let H be a subgroup of G for which (G, H) is a Gel'fand pair and let $H_G = \cap_g H^g$ be the largest normal subgroup contained in H . The probability that the optimal k -register measurement correctly identifies a uniformly random conjugate of H is*

$$P_{\text{success}} \leq \frac{1}{|C(H)|} \cdot \left(\frac{|H|}{|H_G|} \right)^k \quad (3.4)$$

Proof. Without sacrificing optimality, we may initially carry out weak Fourier sampling, in which we observe a representation σ with probability

$$P(\sigma) = \frac{|H|^k d_\sigma \text{rk} \pi_g^\sigma}{|G|^k} .$$

Again accounting for the fact that $\rho_g = \rho_{g'}$ when $H^g = H^{g'}$, the probability of success may then be written

$$P_{\text{success}} = |\text{Norm}(H)| \cdot \text{Exp}_g \text{tr } E_g \rho_g = |\text{Norm}(H)| \cdot \text{Exp}_g \text{Exp}_\sigma a_\sigma \text{tr } E_g^\sigma \rho_g^\sigma$$

where σ is distributed according to $P(\sigma)$, g is uniform in G , and $a_\sigma = 1/\text{tr } \rho_g^\sigma$ normalizes ρ_g^σ so that it is a density matrix.

Recall that ρ_g^σ is proportional to $\pi_g^\sigma \otimes \mathbb{1}_{d_\sigma}$ where π_g^σ has rank one (or zero), that $M^\sigma = m \otimes \mathbb{1}_{d_\sigma}$, and that the image of π_g^σ is contained in the image of m . Therefore, $m^{-1/2} \pi_g^\sigma m^{-1/2}$ has rank one or zero, and so $E_g^\sigma = m^{-1/2} \pi_g^\sigma m^{-1/2} \otimes \mathbb{1}_{d_\sigma}$ is either zero or a scalar multiple of a projection operator of rank d_σ . By the lemma above, for a representation σ we have $\text{Exp}_g \text{tr } (a_\sigma E_g^\sigma \rho_g^\sigma) \leq d_\sigma/|G|$.

Let us again define $S_{H^k} \subset \widehat{G^k}$ to be the set of representations for which π_g^σ is nonzero. By commuting the two expectations above, we conclude that

$$P_{\text{success}} \leq |\text{Norm}(H)| \text{Exp}_\sigma \frac{d_\sigma}{|G|} = \frac{|\text{Norm}(H)|}{|G|} |H|^k \sum_\sigma \text{rk } \pi^\sigma \frac{d_\sigma^2}{|G|^k} = \frac{|H|^k}{|C(H)|} \text{Planch}(S_{H^k})$$

where $\text{Planch}_{G^k}(\sigma) = d_\sigma^2/|G|^k$ is the Plancherel distribution on $\widehat{G^k}$. But this is just the product of the Plancherel distribution on $\widehat{G}$ over the σ_i , so we have

$$P_{\text{success}} \leq \frac{|H|^k}{|C(H)|} \text{Planch}(S_H)^k$$

and recalling from the proof of Theorem 4 that $\text{Planch}(S_H) \leq 1/|H_G|$ completes the proof. $\square$

3.2 Examples

Theorem 5 applies to a number of group families that have appeared in the literature on the hidden subgroup problem. Here is a short list of examples of Gel'fand pairs:

- (G, H) where H is normal and G/H is Abelian. Of course, whenever H is normal the hidden conjugate problem becomes trivial.
- (D_n, H) where H consists of the identity and an involution, as in Bacon, Childs and van Dam [1].
- $(A_p, \mathbb{Z}_p^*)$ where A_p is the affine group $\mathbb{Z}_p^* \ltimes \mathbb{Z}_p$ and $\mathbb{Z}_p^*$ is a maximal non-normal subgroup. An efficient quantum algorithm for the hidden conjugate problem in this case was given by [16].
- All the subgroups of the Heisenberg group, for which an information-theoretic reconstruction algorithm was given by Radhakrishnan, Rötteler and Sen [17].
- $(SL_2(q), B)$ or $(GL_2(q), B)$ where B is the Borel subgroup consisting of upper-triangular matrices.
- (S_n, H) where H is the *hyperoctahedral group*; this is the centralizer of $(1\ 2)(3\ 4) \cdots (n-1\ n)$, or equivalently the wreath product $S_{n/2} \wr \mathbb{Z}_2$, or the symmetry group of the $(n/2)$ -dimensional hyperoctahedron.
- (S_n, H) where $H = S_m \times S_{n-m}$ for some $0 \leq m \leq n$, i.e., the subgroup of permutations under which the set consisting of the first m elements is invariant.

Note there is an efficient classical algorithm for the hidden conjugate problem for the last two examples in S_n : simply check for all $\binom{n}{2}$ transpositions whether the oracle differs from its value on the identity. This allows us to determine the conjugate of H , which is associated with a matching (for the hyperoctahedral group) or a subset of size m (for $S_m \times S_n$).

4 Conclusion

The hidden conjugate problem has important applications; in the dihedral group it is related to hidden shift problems [3] and lattice problems [18]. However, for problems such as Graph Isomorphism, we are typically interested in distinguishing one conjugacy class from another. While we can detect the trivial subgroup with the additional measurement M_0 defined here and in [1], the PGM is not generally optimal in this case [D. Bacon, personal communication]. Constructing the optimal measurement for the hidden subgroup problem, given a prior on the conjugacy classes, remains an important open question.

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