

ON THE EQUATIONAL DEFINABILITY OF BROUWER-ZADEH LATTICES

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ABSTRACT. We give an axiomatisation of the variety of Brouwer-Zadeh lattices, suitable for applications to quantum theory.

1. INTRODUCTION

A *Brouwer-Zadeh poset* (resp. *lattice*) is a structure $\langle A; \leq, \neg, \sim, 0, 1 \rangle$ of type $\langle 2, 1, 1, 0, 0 \rangle$ where $\langle A; \leq, 0, 1 \rangle$ is a bounded poset (resp. bounded lattice), such that the following universal sentences are satisfied:

$$\neg\neg x \approx x \tag{1}$$

$$x \leq y \supset \neg y \leq \neg x \tag{2}$$

$$x \wedge \neg x \leq y \vee \neg y \tag{3}$$

$$x \leq \sim\sim x \tag{4}$$

$$x \leq y \supset \sim y \leq \sim x \tag{5}$$

$$x \wedge \sim x \approx \mathbf{0} \tag{6}$$

$$\neg\sim x \approx \sim\sim x. \tag{7}$$

(Here for readability we are omitting all quantification of variables.)

Brouwer-Zadeh posets (resp. lattices) were introduced by Cattaneo and Marino [2] in connection with unsharp approaches to quantum theory. Given a Hilbert space \mathcal{H} , the set $E(\mathcal{H})$ of all effects can be naturally structured as a Brouwer-Zadeh poset [3]. Moreover, every Brouwer-Zadeh poset embeds as a Brouwer-Zadeh poset into a complete Brouwer-Zadeh lattice, namely its MacNeille completion [5]. Brouwer-Zadeh lattices and their corresponding logics have since been studied by a variety of authors, including Cattaneo and Nisticó [3], Giuntini [6, 7] and Cattaneo and Ciucci [1]. For a survey of quantum logics and

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their associated algebraic semantics, including Brouwer-Zadeh lattices, see Dalla Chiara and Giuntini [4].

When the partial ordering of a Brouwer-Zadeh poset \mathbf{A} is a lattice ordering, no harm comes from systematically confusing \mathbf{A} with its underlying algebra $\langle A; \wedge, \vee, \neg, \sim, 0, 1 \rangle$. (Here $\langle A; \wedge, \vee \rangle$ is the lattice reduct of \mathbf{A} .) In consequence, we may view a Brouwer-Zadeh lattice as an algebra $\langle A; \wedge, \vee, \neg, \sim, 0, 1 \rangle$ of type $\langle 2, 2, 1, 1, 0, 0 \rangle$ satisfying the identities and quasi-identities (1)–(7) together with the lattice identities:

$$x \wedge (y \wedge z) \approx (x \wedge y) \wedge z \quad (8)$$

$$x \vee (y \vee z) \approx (x \vee y) \vee z \quad (9)$$

$$x \wedge x \approx x \quad (10)$$

$$x \vee x \approx x \quad (11)$$

$$x \wedge y \approx y \wedge x \quad (12)$$

$$x \vee y \approx y \vee x \quad (13)$$

$$x \wedge (x \vee y) \approx x \quad (14)$$

$$x \vee (x \wedge y) \approx x \quad (15)$$

$$x \wedge \mathbf{0} \approx \mathbf{0} \quad (16)$$

$$x \vee \mathbf{1} \approx \mathbf{1}. \quad (17)$$

(The Brouwer-Zadeh lattice ordering \leq is recovered in the usual way upon setting $x \leq y$ if and only if $x \wedge y \approx x$.) The class **BZ** of all Brouwer-Zadeh lattices is thus a quasivariety. In fact, an elementary lattice-theoretic argument establishes that an algebra $\langle A; \wedge, \vee, \neg, \sim, 0, 1 \rangle$ of type $\langle 2, 2, 1, 1, 0, 0 \rangle$ is a Brouwer-Zadeh lattice if and only if it satisfies the identities (1), (3)–(4) and (6)–(17), together with the identities $\neg(x \vee y) \wedge (\neg x \wedge \neg y) \approx \neg(x \vee y)$ and $\sim(x \vee y) \wedge (\sim x \wedge \sim y) \approx \sim(x \vee y)$. Hence **BZ** is actually equationally definable.

In this note, we present an alternative axiomatisation of the variety of Brouwer-Zadeh lattices, more suited to applications in quantum theory than the equational basis described above. In particular, we establish the following

Theorem. *An algebra $\langle A; \wedge, \vee, \neg, \sim, 0, 1 \rangle$ of type $\langle 2, 2, 1, 1, 0, 0 \rangle$ is a Brouwer-Zadeh lattice if and only if it satisfies the identities (1), (3)–(4) and (6)–(17), together with the identities*

$$\neg(x \vee y) \approx \neg x \wedge \neg y \quad (18)$$

$$\sim(x \vee y) \approx \sim x \wedge \sim y. \quad (19)$$

Hence the class of all Brouwer-Zadeh lattices is a variety.

2. THE PROOF

Let \mathbf{A} be an algebra $\langle A; \wedge, \vee, \neg, \sim, 0, 1 \rangle$ of type $\langle 2, 2, 1, 1, 0, 0 \rangle$ satisfying the identities (1), (3)–(4), (6)–(17) and (18)–(19). Suppose $a \leq b$. Then $b \wedge a = a$, so $b = b \vee (b \wedge a) = b \vee a$, whence $\neg b = \neg(b \vee a) = \neg b \wedge \neg a$ by (18). Thus $\neg b \leq \neg a$. An analogous argument using (19) establishes $\sim b \leq \sim a$. Hence \mathbf{A} is a Brouwer-Zadeh lattice.

In the (machine-oriented) proof of the converse given below, the justification $[i \rightarrow j]$ indicates paramodulation from i into j ; that is, unifying the left-hand side of i with a subterm of j , instantiating j with the corresponding substitution, and replacing the subterm with the corresponding instance of the right-hand side of i . Also, the justification $[i, j]$ indicates an application of modus ponens with major premiss i and minor premiss j . The two identities to be established are derived at steps 77 and 94 respectively. These steps of the proof are flagged with ‘*’ for easy identification.

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|---|-----------------------|
| 1. $\neg(\neg x) \approx x$ | [(1)] |
| 2. $x \wedge y \approx x \supset \neg y \wedge \neg x \approx \neg y$ | [(2)] |
| 3. $x \wedge \sim(\sim x) \approx x$ | [(4)] |
| 4. $x \wedge y \approx x \supset \sim y \wedge \sim x \approx \sim y$ | [(5)] |
| 5. $\neg(\sim x) \approx \sim(\sim x)$ | [(7)] |
| 6. $x \wedge (y \wedge z) \approx (x \wedge y) \wedge z$ | [(8)] |
| 7. $x \vee (y \vee z) \approx (x \vee y) \vee z$ | [(9)] |
| 8. $x \wedge x \approx x$ | [(10)] |
| 9. $x \wedge y \approx y \wedge x$ | [(12)] |
| 10. $x \vee y \approx y \vee x$ | [(13)] |
| 11. $x \wedge (x \vee y) \approx x$ | [(14)] |
| 12. $x \vee (x \wedge y) \approx x$ | [(15)] |
| 13. $\sim(\neg(\sim x)) \approx \neg(\sim(\sim x))$ | [5 \rightarrow 5] |
| 14. $\neg(\sim(\sim x)) \approx \sim x$ | [5 \rightarrow 1] |
| 15. $x \wedge \neg(\sim x) \approx x$ | [5 \rightarrow 3] |
| 16. $(x \wedge y) \wedge y \approx x \wedge y$ | [8 \rightarrow 6] |
| 17. $\neg(\sim x) \vee y \approx y \vee \sim(\sim x)$ | [5 \rightarrow 10] |
| 18. $(x \vee y) \vee z \approx (z \vee x) \vee y$ | [7 \rightarrow 10] |
| 19. $\sim(x \vee y) \wedge \sim x \approx \sim(x \vee y)$ | [4, 11] |
| 20. $\neg(x \vee y) \wedge \neg x \approx \neg(x \vee y)$ | [2, 11] |
| 21. $x \wedge (y \vee x) \approx x$ | [10 \rightarrow 11] |
| 22. $x \vee (y \wedge x) \approx x$ | [9 \rightarrow 12] |
| 23. $(x \wedge y) \vee (x \wedge (y \wedge z)) \approx x \wedge y$ | [6 \rightarrow 12] |
| 24. $x \wedge (y \wedge \neg(\sim(x \wedge y))) \approx x \wedge y$ | [6 \rightarrow 15] |

25. $\sim(x \vee y) \wedge \sim y \approx \sim(x \vee y)$	[4, 21]
26. $\neg(x \vee y) \wedge \neg y \approx \neg(x \vee y)$	[2, 21]
27. $\neg(\sim x) \vee x \approx \neg(\sim x)$	[15 \rightarrow 22]
28. $\sim(\neg(\sim x)) \approx \sim x$	[14 \rightarrow 13]
29. $\sim x \wedge \sim(y \wedge x) \approx \sim x$	[4, 16]
30. $\neg x \wedge \neg(y \wedge x) \approx \neg x$	[2, 16]
31. $x \vee \neg(\sim x) \approx \neg(\sim x)$	[10 \rightarrow 27]
32. $\sim x \wedge \sim(y \wedge \neg(\sim x)) \approx \sim(\neg(\sim x))$	[28 \rightarrow 29]
33. $\sim x \wedge \sim(x \wedge y) \approx \sim x$	[9 \rightarrow 29]
34. $\sim(x \wedge y) \vee \sim y \approx \sim(x \wedge y)$	[29 \rightarrow 22]
35. $\neg x \wedge \neg(x \wedge y) \approx \neg x$	[9 \rightarrow 30]
36. $\neg(x \wedge y) \vee \neg y \approx \neg(x \wedge y)$	[30 \rightarrow 22]
37. $\sim(x \wedge y) \vee \sim x \approx \sim(x \wedge y)$	[33 \rightarrow 22]
38. $\neg(x \wedge y) \vee \neg x \approx \neg(x \wedge y)$	[35 \rightarrow 22]
39. $\sim(x \vee y) \wedge \sim y \approx \sim(y \vee x)$	[10 \rightarrow 19]
40. $\neg x \wedge \neg(x \vee y) \approx \neg(x \vee y)$	[9 \rightarrow 20]
41. $\neg x \wedge \neg(y \vee x) \approx \neg(y \vee x)$	[9 \rightarrow 26]
42. $\sim x \vee \sim(y \wedge x) \approx \sim(y \wedge x)$	[10 \rightarrow 34]
43. $\neg x \vee \neg(y \wedge x) \approx \neg(y \wedge x)$	[10 \rightarrow 36]
44. $\sim(\sim x \wedge y) \vee \neg(\sim x) \approx \sim(\sim x \wedge y)$	[5 \rightarrow 37]
45. $\neg x \vee \neg(x \wedge y) \approx \neg(x \wedge y)$	[10 \rightarrow 38]
46. $\sim(x \vee y) \approx \sim(y \vee x)$	[25 \rightarrow 39]
47. $\sim x \wedge \sim(y \vee x) \approx \sim(x \vee y)$	[9 \rightarrow 39]
48. $\sim(\neg(\sim x) \vee y) \approx \sim(y \vee \sim(\sim x))$	[5 \rightarrow 46]
49. $x \wedge \neg(\neg x \vee y) \approx \neg(\neg x \vee y)$	[1 \rightarrow 40]
50. $\neg x \wedge \neg(\neg(\sim x)) \approx \neg(x \vee \neg(\sim x))$	[31 \rightarrow 40]
51. $x \wedge \neg(y \vee \neg x) \approx \neg(y \vee \neg x)$	[1 \rightarrow 41]
52. $(x \wedge \sim y) \vee (x \wedge \sim(y \vee z)) \approx x \wedge \sim y$	[47 \rightarrow 23]
53. $\sim(\sim(\sim x) \vee \sim(\sim y)) \approx \sim(\neg(\sim y) \vee \neg(\sim x))$	[5 \rightarrow 48]
54. $\sim x \wedge \neg(\neg(\sim x) \vee y) \approx \neg(\sim(\sim x) \vee y)$	[5 \rightarrow 49]
55. $(x \wedge y) \vee (x \wedge \neg(\neg y \vee z)) \approx x \wedge y$	[49 \rightarrow 23]
56. $x \vee \neg(\neg x \vee y) \approx x$	[49 \rightarrow 12]
57. $x \vee \neg(y \vee \neg x) \approx x$	[10 \rightarrow 56]
58. $\sim x \wedge \sim(y \wedge \neg(\sim x)) \approx \sim x$	[28 \rightarrow 32]
59. $\sim x \wedge \sim(\neg(\sim x \vee y)) \approx \sim x$	[20 \rightarrow 58]
60. $\neg(x \vee \neg(\sim x)) \approx \neg x \wedge \sim x$	[1 \rightarrow 50]
61. $\neg x \wedge \sim x \approx \neg(\neg(\sim x))$	[31 \rightarrow 60]
62. $\neg x \wedge \sim x \approx \sim x$	[1 \rightarrow 61]
63. $(x \wedge y) \vee \neg(\neg y \vee \neg x) \approx x \wedge y$	[51 \rightarrow 55]
64. $(\neg x \vee \neg y) \vee \neg(y \wedge x) \approx \neg x \vee \neg y$	[63 \rightarrow 57]
65. $(x \wedge \sim(y \vee z)) \vee (x \wedge \sim y) \approx x \wedge \sim y$	[10 \rightarrow 52]
66. $\neg(\sim(\sim x) \vee y) \approx \neg(\neg(\sim x) \vee y)$	[49 \rightarrow 54]

67.	$\neg(\sim(\sim x) \vee y) \approx \neg(y \vee \neg(\sim x))$	[10 → 66]
68.	$\neg x \vee (\neg y \vee \neg(y \wedge x)) \approx \neg x \vee \neg y$	[7 → 64]
69.	$\sim(x \vee y) \vee (\neg(x \vee y) \wedge \sim x) \approx \neg(x \vee y) \wedge \sim x$	[62 → 65]
70.	$\neg x \vee \neg(y \wedge x) \approx \neg x \vee \neg y$	[45 → 68]
71.	$\neg x \vee \neg y \approx \neg(y \wedge x)$	[43 → 70]
72.	$\neg(x \wedge \neg y) \approx y \vee \neg x$	[1 → 71]
73.	$\neg(\neg x \wedge y) \approx \neg y \vee x$	[1 → 71]
74.	$\neg(x \vee \neg y) \approx y \wedge \neg x$	[72 → 1]
75.	$\neg(\sim(\sim x) \vee y) \approx \sim x \wedge \neg y$	[67 → 74]
76.	$\neg(x \vee \neg y) \approx \neg x \wedge y$	[9 → 74]
*77.	$\neg x \wedge \neg y \approx \neg(x \vee y)$	[1 → 76]
78.	$\sim x \wedge \neg(\sim(y \wedge \sim x)) \approx \neg(\sim(y \wedge \sim x))$	[42 → 75]
79.	$x \wedge \neg(\sim(x \wedge \sim y)) \approx x \wedge \sim y$	[78 → 24]
80.	$\sim(x \wedge \sim y) \vee \neg x \approx \neg(x \wedge \sim y)$	[79 → 72]
81.	$\sim(\sim x \wedge \sim y) \approx \neg(\sim x \wedge \sim y)$	[44 → 80]
82.	$\sim(\neg(\sim x \wedge \sim y)) \approx \neg(\sim(\sim x \wedge \sim y))$	[81 → 5]
83.	$\neg(\sim(\sim x \wedge \sim y)) \approx \sim x \wedge \sim y$	[81 → 1]
84.	$\sim(\neg(\sim x \wedge \sim y)) \approx \sim x \wedge \sim y$	[83 → 82]
85.	$\sim(\sim(\sim x) \vee \sim(\sim y)) \approx \sim(\neg(\sim x \wedge \sim y))$	[71 → 53]
86.	$\sim(\sim(\sim x) \vee \sim(\sim y)) \approx \sim x \wedge \sim y$	[84 → 85]
87.	$\sim(x \vee y) \wedge \sim(\neg(\neg(x \vee y) \wedge \sim x)) \approx \sim(x \vee y)$	[69 → 59]
88.	$\sim(x \vee y) \wedge \sim(\neg(\sim x) \vee (x \vee y)) \approx \sim(x \vee y)$	[73 → 87]
89.	$\sim((x \vee y) \vee \neg(\sim x)) \approx \sim(x \vee y)$	[47 → 88]
90.	$\sim((\neg(\sim x) \vee x) \vee y) \approx \sim(x \vee y)$	[18 → 89]
91.	$\sim(\neg(\sim x) \vee y) \approx \sim(x \vee y)$	[27 → 90]
92.	$\sim(x \vee \sim(\sim y)) \approx \sim(y \vee x)$	[17 → 91]
93.	$\sim(x \vee \sim(\sim y)) \approx \sim y \wedge \sim x$	[86 → 92]
*94.	$\sim x \wedge \sim y \approx \sim(x \vee y)$	[92 → 93]

The preceding proof was obtained with the assistance of the automated reasoning program Otter [8], using the method of proof sketches [10]. For examples of the application of automated reasoning to a wide range of problems in equational logic, see [9].

REFERENCES

- [1] G. Cattaneo and D. Ciucci, *Algebraic structures for rough sets*, LNCS Transactions on Rough Sets **3135** (2004), 218–264.
- [2] G. Cattaneo and G. Marino, *Brouwer-Zadeh posets and fuzzy set theory*, Proc. Mathematics of Fuzzy Systems (Napoli) (A. Di Nola and A. Ventre, eds.), 1984, pp. 35–84.
- [3] G. Cattaneo and G. Nisticó, *Brouwer-Zadeh posets and three-valued Lukasiewicz posets*, Fuzzy Sets and Systems **33** (1989), 165–190.

- [4] M. L. Dalla Chiara and R. Giuntini, *Quantum logics*, Handbook of Philosophical Logic (D. Gabbay and F. Guentner, eds.), vol. VI, Kluwer Academic Publishers, Dordrecht, 2002, pp. 129–228.
- [5] R. Giuntini, *A semantical investigation of Brouwer-Zadeh logic*, J. Philos. Logic **20** (1991), 411–433.
- [6] ———, *Brouwer-zadeh logic, decidability and bimodal systems*, Studia Logica **51** (1992), 97–112.
- [7] ———, *Three-valued Brouwer-Zadeh logic*, Internat. J. Theoret. Physics **32** (1993), 1875–1887.
- [8] W. McCune, *Otter 3.3 Reference Manual*, Tech. Memo ANL/MCS-TM-263, Mathematics and Computer Science Division, Argonne National Laboratory, Argonne, IL, August 2003.
- [9] W. McCune and R. Padmanabhan, *Automated deduction in equational logic and cubic curves*, Lecture Notes in Computer Science (AI subseries), vol. 1095, Springer-Verlag, Berlin, 1996.
- [10] R. Veroff, *Solving open questions and other challenge problems using proof sketches*, J. Automated Reasoning **27** (2001), no. 2, 157–174.

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