

IS $\mathbf{H}(\mathbf{BCK})$ FINITELY BASED?

A *BCK-algebra* is an algebra $\langle A; *, 0 \rangle$ of type $\langle 2, 0 \rangle$ satisfying the following collection of identities and quasi-identities:

$$\begin{aligned} ((x * y) * (x * z)) * (z * y) &\approx \mathbf{0} \\ x * \mathbf{0} &\approx x \\ \mathbf{0} * x &\approx \mathbf{0} \end{aligned}$$

$$x * y \approx \mathbf{0} \ \& \ y * x \approx \mathbf{0} \supset x \approx y.$$

By a result of Wroński, the class \mathbf{BCK} of all BCK-algebras is not a variety. The varietal closure of \mathbf{BCK} is $\mathbf{H}(\mathbf{BCK})$, the class of all homomorphic images of members of \mathbf{BCK} .

Problem: Is $\mathbf{H}(\mathbf{BCK})$ finitely based? If $\mathbf{H}(\mathbf{BCK})$ is finitely based, provide a finite axiomatisation.

Very recently, Andrzej Wroński has conjectured that $\mathbf{H}(\mathbf{BCK})$ is finitely based, with equational basis

$$\mathbf{0} * x \approx \mathbf{0} \tag{1}$$

$$((x * y) * (z * y)) * (x * z) \approx \mathbf{0} \tag{2}$$

$$x * \mathbf{0} \approx x \tag{3}$$

$$x * (x * ((x * y) * z)) \approx (x * y) * z \tag{4}$$

$$(x * y) * z \approx (x * z) * y. \tag{5}$$

Recent work with R. J. Bignall and R. Veroff on the poset structure of BCK-algebras suggests Wroński's proposed equational basis is very plausible.

Problem: Prove Wroński's conjecture, or find a counterexample.

Let \mathbf{H} denote the variety with equational basis given by (1)–(5) above. By standard arguments on BCK-algebras, to prove Wroński's conjecture it would be sufficient to verify the following

Conjecture: If $\mathbf{H} \models (x * \Gamma) * (x * \Delta) \approx \mathbf{0}$ and $\mathbf{H} \models (x * \Delta) * (x * \Gamma) \approx \mathbf{0}$ then $\mathbf{H} \models x * \Delta \approx x * \Gamma$.

Here Γ, Δ denote multisets of all antecedents a_0, \dots, a_n , for any expression of the form $((\dots (b * a_n) \dots) * a_1) * a_0$.