A Variety Containing Quasi-Hilbert Algebras

Michael K. Kinyon

Department of Mathematics
University of Denver

Automated Deduction and its Application to Mathematics
2007

Definitions

Hilbert and quasi-Hilbert algebras are algebraic models of the implicational fragments of certain sentential logics.
Common axioms:

$$x \cdot x = 1 \tag{A1}$$

$$x \cdot (y \cdot x) = 1 \tag{A2}$$

$$x \cdot y = 1 \wedge y \cdot x = 1 \Rightarrow x = y \tag{A3}$$

For *quasi-Hilbert algebras*, an additional axiom:

$$x \cdot (y \cdot z) = 1 \wedge x \cdot (y \cdot (z \cdot u)) = 1 \Rightarrow x \cdot (y \cdot u) = 1 \tag{QA}$$

For *Hilbert algebras*, replace (QA) with:

$$(x \cdot (y \cdot z)) \cdot ((x \cdot y) \cdot (x \cdot z)) = 1 \tag{HA}$$

(and then (A1) is dependent)

Hilbert Algebras . . .

- model the implicational fragment of intuitionistic logic
- form a variety (Diego 1965)
- ... which is not generated by any of its finite members (Celani & Cabrer 2005)

Equational basis:

$$x \cdot x = 1 \qquad 1 \cdot x = x \qquad x \cdot (y \cdot z) = (x \cdot y) \cdot (x \cdot z)$$
$$(x \cdot y) \cdot ((y \cdot x) \cdot x) = (y \cdot x) \cdot ((x \cdot y) \cdot y)$$

Quasi-Hilbert Algebras . . .

- model certain logics and Gentzen systems
- form a quasi-variety (clear from axioms)

Every Hilbert algebra is a quasi-Hilbert algebra.

Quasi-Hilbert Algebras . . .

- model certain logics and Gentzen systems
- form a quasi-variety (clear from axioms)

Every Hilbert algebra is a quasi-Hilbert algebra.

(Proof: if not, the latter would not be called “quasi-Hilbert”.)

$:-)$

A NonHilbert, Quasi-Hilbert Algebra

$\cdot$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	1	1	3	4	6	6
3	1	2	1	5	5	6
4	1	1	1	1	1	1
5	1	1	3	3	1	1
6	1	2	3	4	2	1

Table: The Smallest NonHilbert, Quasi-Hilbert Algebra

The Problem

Problem

Do Quasi-Hilbert Algebras Form A Variety?

. . . If so, find a (hopefully finite) equational basis.

This problem was

- perhaps posed by Diego (1965, in Catalan!)
- certainly posed by Bou *et al* (2004)
- brought to my attention by Matthew Spinks

What Makes This Hard

If quasi-Hilbert algebras form a variety, the difficulty is finding equations to replace this axiom:

$$x \cdot (y \cdot z) = 1 \wedge x \cdot (y \cdot (z \cdot u)) = 1 \Rightarrow x \cdot (y \cdot u) = 1 \qquad \text{(QA)}$$

(Replacing (A3) is quite easy.)

An Attack on the Problem

Let **EQ** $= \{(\mathrm{A1}), (\mathrm{A2})\}$, the equational axioms for quasi-Hilbert algebras.

while (the following works) do {

- Use `Mace4` to generate a model in which **EQ** holds, but (QA) is false.
- Use `Prover9` semantics to generate equations true in quasi-Hilbert algebras, but false in the model.
- Enlarge **EQ** by the new equations.
- Reduce **EQ** by getting rid of dependencies.
- Check if **EQ** now implies (QA). If so, quit.

}

. . . And Where It Stops, Nobody Knows. . .

Of course, I didn't find a set **EQ** which implies (QA). Eventually, a 10-element model was found such that every equation `Prover9` could generate which was true in quasi-Hilbert algebras is also true in the model.

Axioms for a New Variety

$$1 \cdot x = x \tag{NA1}$$

$$x \cdot (x \cdot y) = x \cdot y \tag{NA2}$$

$$(((x \cdot y) \cdot y) \cdot z) \cdot (x \cdot z) = 1 \tag{NA3}$$

$$((x \cdot y) \cdot z) \cdot (y \cdot ((z \cdot u) \cdot u)) = 1 \tag{NA4}$$

$$(((x \cdot y) \cdot y) \cdot x) \cdot ((x \cdot y) \cdot y) = (((x \cdot y) \cdot y) \cdot x) \cdot x \tag{NA5}$$

A Non-Quasi-Hilbert Algebra Satisfying (NA1)-(NA5)

$\cdot$	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	1	1	3	7	5	1	7	10	1	10
3	1	2	1	6	5	6	1	9	9	1
4	1	2	3	1	8	1	1	8	9	10
5	1	2	3	1	1	1	1	1	1	1
6	1	2	3	7	5	1	7	8	9	10
7	1	2	3	6	5	6	1	8	9	10
8	1	2	3	4	4	6	7	1	1	1
9	1	2	3	4	5	6	7	10	1	10
10	1	2	3	4	5	6	7	9	9	1

Is This Interesting?

This would not be an interesting variety, except. . .

Many "logical" theorems true in quasi-Hilbert algebras are true in this variety.

Firstly, these axioms of quasi-Hilbert algebras are theorems in the new variety:

$$x \cdot x = 1 \tag{A1}$$

$$x \cdot (y \cdot x) = 1 \tag{A2}$$

$$x \cdot y = 1 \wedge y \cdot x = 1 \Rightarrow x = y \tag{A3}$$

More Theorems

$$x \cdot 1 = 1 \quad \text{(Pre)}$$

$$x \cdot y = 1 \;\wedge\; y \cdot z = 1 \;\Rightarrow\; x \cdot z = 1 \quad \text{(Trans)}$$

$$x \cdot y = 1 \;\wedge\; x \cdot (y \cdot z) = 1 \;\Rightarrow\; x \cdot z = 1 \quad \text{(MP1)}$$

$$x \cdot y = 1 \;\Rightarrow\; (y \cdot z) \cdot (x \cdot z) = 1 \quad \text{(Isot1)}$$

$$x \cdot y = 1 \;\Rightarrow\; (z \cdot x) \cdot (z \cdot y) = 1 \quad \text{(Isot2)}$$

$$x \cdot y = 1 \;\wedge\; z \cdot u = 1 \;\Rightarrow\; (y \cdot z) \cdot (x \cdot u) = 1 \quad \text{(Cong)}$$

$$x \cdot (y \cdot z) = 1 \;\Rightarrow\; y \cdot (x \cdot z) = 1 \quad (\mathrm{Q}_{CP})$$

$$x \cdot (y \cdot z) = 1 \;\Leftrightarrow\; (x \cdot y) \cdot (x \cdot z) = 1 \quad (\mathrm{Q}_{Fre1})$$

$$(x \cdot y) \cdot ((y \cdot x) \cdot x) = 1 \;\Rightarrow\; (y \cdot x) \cdot ((x \cdot y) \cdot y) = 1 \quad (\mathrm{Q}_{Fre2})$$

A New Attack

- Use (NA*j*) and (HA) to generate equations true in Hilbert algebras, but false in the 10-element model
- For each new equation α, check if (NA*j*) $+\alpha \Rightarrow$ (HA)
- If not, check if α is a theorem in quasi-Hilbert algebras. If so, add it to **EQ**.

Bleah!

But it didn't work. . .

Every equation `Prover9` generated (until I killed the jobs) which was false in the model turned out be equivalent to (HA) within the new variety.

Other Equations

In the course of various experiments, other equations turned up which are

- theorems in Hilbert algebras
- true in the non-quasi-Hilbert, 10-element model of (NA*j*)
- false in a 10-element, nonHilbert, quasi-Hilbert algebra
- true in a different 10-element, nonHilbert, quasi-Hilbert algebra

Example: $x \cdot (((y \cdot z) \cdot (x \cdot y)) \cdot ((y \cdot z) \cdot ((z \cdot u) \cdot u))) = 1$

Non-quasi-Hilbert models of (NA*j*) in which these equations are false might be useful.

Commutative Hilbert Algebras

Jun (1996) defined a Hilbert algebra to be *commutative* if

$$(x \cdot y) \cdot y = (y \cdot x) \cdot x$$

(This means the operation $x \vee y = (x \cdot y) \cdot y$ is commutative.) Halaš (2002) showed that commutative Hilbert algebras are exactly *Abbott implication algebras*.

(Petr and RP talked about such algebras last year.)

Abbott implication algebras

Axioms for Abbott implication algebras:

$$x \cdot x = 1$$
$$(x \cdot y) \cdot x = x$$
$$x \cdot (y \cdot z) = y \cdot (x \cdot z)$$
$$(x \cdot y) \cdot y = (y \cdot x) \cdot x$$

Halaš's proof is roundabout: he shows that the operation $x \vee y = (x \cdot y) \cdot y$ in a commutative Hilbert algebra gives a join semilattice, and then appeals to a theorem of Abbott (1967) to reach the desired conclusion.

A Generalization

Within a few seconds, `Prover9` can get a syntactic proof of

Theorem

An algebra $(Q; \cdot, 1)$ satisfying (NA1), (NA2), (NA4) and $(x \cdot y) \cdot y = (y \cdot x) \cdot x$ is an Abbott implication algebra.

So, for instance, commutative quasi-Hilbert algebras are Abbott implication algebras.

The Natural Order

In an algebra satisfying (NAj), $j = 1, \ldots, 5$, define a relation by

$$x \leq y \qquad \Leftrightarrow \qquad x \cdot y = 1$$

This is a partial order with unique maximal element 1.

(In any ordered set with maximal element 1, the operation given by

$$x \cdot y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}$$

defines a Hilbert algebra structure.)

The Hilbert Algebra Case

Figallo *et al* defined a *Hilbert algebra with Infimum* to be a Hilbert algebra such that the associated poset is a meet semilattice.

Characterization as a variety of algebras $(Q; \cdot, \wedge, 1)$:

- $(Q; \cdot, 1)$ is a Hilbert algebra,
- $(Q; \wedge)$ is a semilattice,

and these equations hold:

$$x \wedge (x \cdot y) = x \wedge y \qquad \text{(WA1)}$$

$$(x \cdot (y \wedge z)) \cdot ((x \cdot y) \wedge (x \cdot z)) = 0 \qquad \text{(WA2)}$$

(The smallest Hilbert algebra without infima is the obvious one of size 3.)

Definition and New Problem

Define a *quasi-Hilbert algebra with Infimum* to be a quasi-Hilbert algebra such that the associated poset is a meet semilattice.

(The 6-element non-Hilbert, quasi-Hilbert algebra has infima. The smallest non-Hilbert, quasi-Hilbert algebra without infima has size 7.)

Problem

Do quasi-Hilbert algebras with infimum form a variety?

Yes!

Theorem

An algebra $(Q; \cdot, \wedge, 1)$ *is a quasi-Hilbert algebra with infimum if and only if:*

- $(Q; \cdot, 1)$ *satisfies (NAj),* $j = 1, \ldots, 5$*,*
- $(Q; \wedge)$ *is a semilattice,*

and these equations hold:

$$x \wedge (x \cdot y) = x \wedge y \qquad \text{(WA1)}$$

$$(x \cdot (y \wedge z)) \cdot ((x \cdot y) \wedge (x \cdot z)) = 0 \qquad \text{(WA2)}$$

$$((x \cdot y) \wedge (x \cdot (y \cdot z))) \cdot (x \cdot z) = 0 \qquad \text{(WA3)}$$