FOUR PROBLEMS IN THE EQUATIONAL THEORY OF n-POTENT BCK-ALGEBRAS

A *BCK-algebra* is an algebra $\langle A; *, 0\rangle$ of type $\langle 2, 0\rangle$ satisfying the following collection of identities and quasi-identities:

$$((x * y) * (x * z)) * (z * y) \approx \mathbf{0}$$
$$x * \mathbf{0} \approx x$$
$$\mathbf{0} * x \approx \mathbf{0}$$

$$x * y \approx \mathbf{0} \,\&\, y * x \approx \mathbf{0} \supset x \approx y.$$

By a result of Wroński, the class BCK of all BCK-algebras is not a variety.

For any integer $n \geq 1$, consider the terms $x * y^n$ defined inductively by

$$x * y^0 := x$$
$$x * y^{k+1} := (x * y^k) * y \qquad \text{for } k \geq 1.$$

A class of BCK-algebras is said to be $n+1$-*potent*, $1 \leq n \in \omega$, if it satisfies the identity:

$$xy^{n+1} \approx xy^n \tag{E_n}$$

For each integer $n \geq 1$, the class of all BCK-algebras satisfying (E_n) is equationally definable. Every finite BCK-algebra satisfies (E_n) for some $n \geq -1$.

An *implicative BCK-algebra* is a BCK-algebra satisfying the identity

$$x * (y * x) \approx x.$$

Let $\mathbf{A}$ be a BCK-algebra satisfying (E_1). For all $a, b \in A$, let:

$$a/b := a * \big((a * (a * b)) * (b * a)\big).$$

By a result of Guzmán, the term reduct $\langle A; /, 0\rangle$ is an implicative BCK-algebra.

Problem: Let $\mathbf{A}$ be a BCK-algebra satisfying (E_n). For all $a, b \in A$, let:

$$a \cap b := \big((a * (a * b)^n) * (b * a)^n\big)$$
$$a/b := a * (a \cap b).$$

Is the term reduct $\langle A; \cap, 0\rangle$ a meet semilattice with zero?
Is the term reduct $\langle A; /, 0\rangle$ an implicative BCK-algebra?

A BCK-algebra $\mathbf{A}$ is said to be *right ideal commutative* if it satisfies the identity:

$$(z * x) * \big((z * x) * (z * y)\big) \approx (z * y) * \big((z * y) * (z * x)\big).$$

The class of all right ideal commutative BCK-algebras is *not* equationally definable.

A BCK-algebra $\mathbf{A}$ is said to have *Condition (J)* if it satisfies the identity:

$$x * \big(x * (y * (y * x))\big) \approx y * \big(y * (x * (x * y))\big).$$

Clearly the class of all BCK-algebras having Condition (J) is a variety.

For each $n \in \{-1\} \cup \omega$, define the term $j_n(x, y)$ inductively by

$$j_{-1}(x, y) := x,$$
$$j_{2n}(x, y) := y * \big(y * (j_{2n-1}(x, y))\big),$$
$$j_{2n+1}(x, y) := x * \big(x * (j_{2n}(x, y))\big).$$

and consider the identity:

$$j_n(x, y) \approx j_n(y, x). \qquad (\mathrm{J}_n)$$

Clearly each (J_n), $n \geq 1$, generalises Condition (J). The class of all BCK-algebras satisfying (J_n) is a variety for each $n \in \{-1\} \cup \omega$. Moreover, every finite BCK-algebra satisfies (J_n) for some $n \geq -1$.

Problem: Let $\mathbf{A}$ be a right ideal commutative BCK-algebra. Suppose that $\mathbf{A} \models (\mathrm{E}_n)$ for any integer $n \geq 1$. Is it the case that $\mathbf{A} \models (\mathrm{J}_n)$?

It is quite plausible that the proceding problem has a positive solution. Assume for the sake of argument that it has. Let $\mathbf{A}$ be a right ideal commutative BCK-algebra satisfying (E_n) for any integer $n \geq 1$. For all $a, b \in A$, let:

$$a \wedge b := j_n(a, b).$$

Problem: Let $\mathbf{A}$ be a right ideal commutative BCK-algebra such that $\mathbf{A} \models (\mathrm{E}_n)$ for any integer $n \geq 1$. Is the term reduct $\langle A; \wedge, 0\rangle$ a meet semilattice with zero?

Assume for the sake of argument that the preceding problem also has a positive solution. A meet semilattice with zero for which the subalgebra $[\mathbf{0}, \boldsymbol{a}]$ is pseudocomplemented for each $a \in A$ is said to be a *locally pseudocomplemented meet semilattice with zero.* The class of all locally pseudocomplemented meet semilattices with zero is a variety, axiomatised by a set of identities defining meet semilattices with zero together with the identities:

$$x \backslash (x \wedge y) \approx x \backslash y$$
$$x \wedge (y \backslash z) \approx (x \backslash z) \wedge (y \backslash z)$$
$$(x \backslash y) \wedge y \approx \mathbf{0}$$
$$x \backslash \mathbf{0} \approx x.$$

Let $\mathbf{A}$ be a right ideal commutative BCK-algebra satisfying (E_n) for any integer $n \geq 1$. For all $a, b \in A$, let:

$$a - b := a * b^n$$
$$a \backslash b := a - (a \wedge b).$$

Problem: Let $\mathbf{A}$ be a right ideal commutative BCK-algebra such that $\mathbf{A} \models (\mathrm{E}_n)$ for any integer $n \geq 1$. Is the term reduct $\langle A; \wedge, \backslash, 0\rangle$ a locally pseudocomplemented meet semilattice with zero?