ADAM 2011
Banff International Research Station
June 24–26

Dedicated to the memory of Bill McCune

Working Our Way Up a Theory Hierarchy

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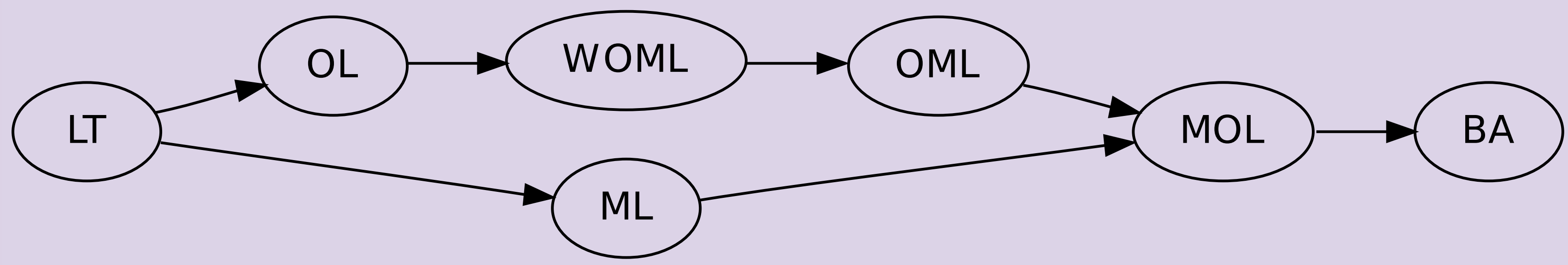
Theory Hierarchies

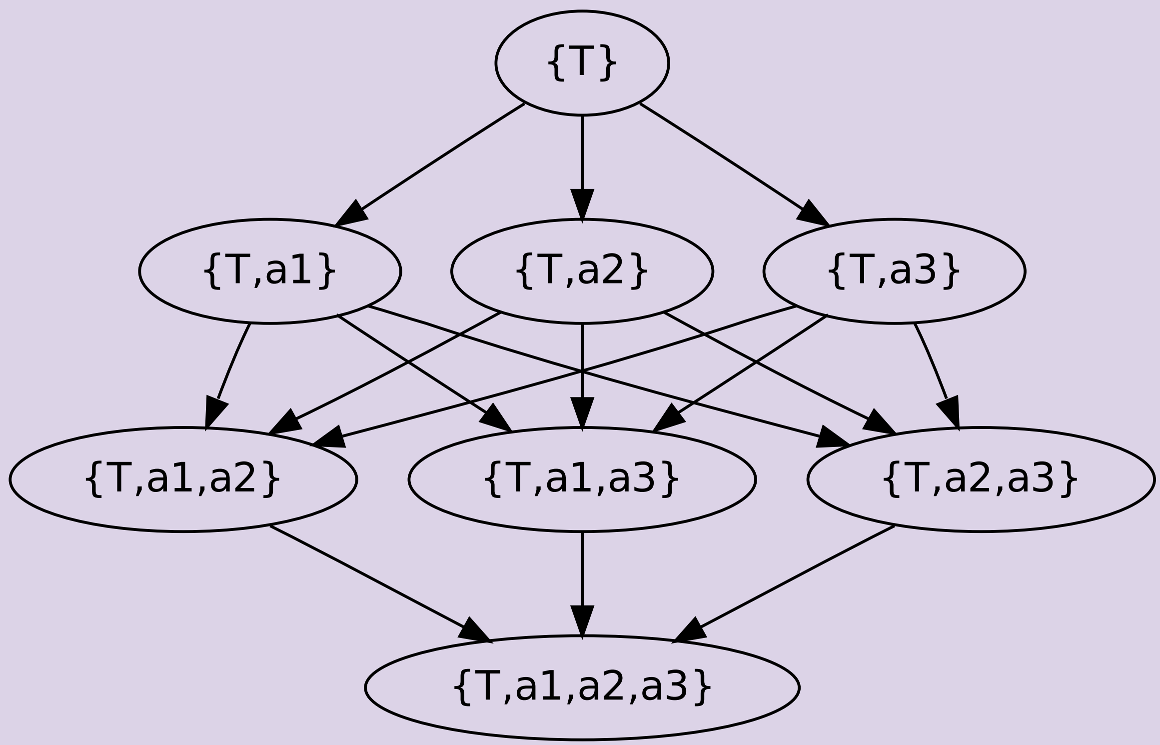
(Example: Lattice Theory)



Scenarios for proving a theorem t

- How far up the hierarchy can we push the result?
- Can we use the hierarchy to help prove t in a specific theory?

Theory Hierarchies



The AIM Problem

- Concerns Abelian inner mappings in loop theory
- Proposed as an automated deduction challenge problem by Michael Kinyon at ADAM 2009
- Collaboration with Petr Vojtěchovský, J. D. Phillips and Aleš Drápal
- Problem description
  - File aim_descr.txt.
  - Several candidate extensions; some combinations of special interest
  - Working our way up the hierarchy

Using the Hierarchy

Let $\mathcal{T}_A$ denote the theorems deducible from axioms A.

If $A^- \subset A \subset A^+$, then $\mathcal{T}_{A^-} \subseteq \mathcal{T}_A \subseteq \mathcal{T}_{A^+}$

independent extra assumptions $\Rightarrow$ proper extensions of the theory

Example: lattice hierarchy

$\mathcal{LT} \subset \mathcal{OL} \subset \mathcal{WOML} \subset \mathcal{OML} \subset \mathcal{MOL} \subset \mathcal{BA}$

Can knowledge about $\mathcal{T}_{A^-}$ and $\mathcal{T}_{A^+}$ help us prove theorems in $\mathcal{T}_A$?

Two ideas for picking clauses (guiding the search)

- semantic guidance (consider models in $\mathcal{T}_{A^-}$)
- proof sketches (consider proofs in $\mathcal{T}_{A^+}$)

Semantic Guidance

- Consider clause sets $C^- \subset C$ and a formula t such that t is a theorem in $\mathcal{T}_\mathcal{C}$ but is *not* a theorem in $\mathcal{T}_{C^-}$.
- Let I be an interpretation (model) for C^- that falsifies t.
- Key observation: A proof of t from C will necessarily include steps that evaluate to **False** under I.
- Idea: Have a selection bias for clauses that evaluate to **False** under I.
- The challenge is to find effective weakenings of the target theory and good candidate interpretations I.

Semantic guidance is supported by Mace4/Prover9.

Proof Sketches

Consider a derivation as a sequence of clauses,

$$c_1, c_2, ..., c_i, ..., c_j, ..., c_n$$

where

- c_i is an extra assumption wrt the target theory ($c_i \in C^+ - C$)
- derived clause c_j has c_i in its derivation history

c_j either is in the target theory or it is not.

- if yes, it suffices to find a new derivation of c_j
- if no, it suffices to "bridge the gaps" wrt the consequences of c_j

In either case, we may have reduced a large problem to a "smaller" (but still potentially difficult) problem.

The Proof Sketches Method

- Idea: Collect proofs of the target theorem in extended theories (i.e., with extra assumptions) and have a selection bias for clauses that subsume clauses in these proofs.
- The emphasis is on the *sufficiency* of the collected “proof sketches”.
- Moving up the hierarchy by systematically generating new proof sketches with fewer extra assumptions, including all previous proof sketches for guidance.
- The challenge is to find effective extensions of the target theory (extra assumptions).

In some sense, the objective is to transform a proof finding problem into a proof completion problem.

Prover9 supports proof sketches via *hints*.

Status of the AIM project

- Summary of results
- Comments about proof lengths

Final comment (for the ADAM 2009 participants) ... p9loop has played a significant role in the AIM project, and I've gotten much better at "unwinding" the proofs.