Subloops of size 32 of Cayley-Dickson Loops

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Cayley-Dickson doubling process

Define a sequence of power-associative algebras inductively

$$\mathbb{A}_0 = \mathbb{R},\ \ a^* = a,\ \text{ where } a \in \mathbb{R}$$
$$\mathbb{A}_n = \{(a, b) \mid a, b \in A_{n-1}\},\ \ n \in \mathbb{N}$$

with multiplication $(a, b) \cdot (c, d) = (a \cdot c - d^* \cdot b, d \cdot a + b \cdot c^*)$
addition $(a, b) + (c, d) = (a + c, b + d)$
and conjugation $(a, b)^* = (a^*, -b)$

Theorem (Hurwitz, 1898)

The only normed division algebras over $\mathbb{R}$ are $\mathbb{R}$ (real numbers), $\mathbb{C}$ (complex numbers), $\mathbb{H}$ (quaternions) and $\mathbb{O}$ (octonions).

Cayley-Dickson loops

Definition

Define **Cayley-Dickson loops** $(Q_n, \cdot)$ inductively

$$\begin{aligned} Q_0 &= \mathbb{R}_2 = \{1, -1\} \\ Q_n &= \{(x,0), (x,1), \quad x \in Q_{n-1}\} \end{aligned}$$

For example

$$\begin{aligned} Q_1 &= \mathbb{C}_4 = \pm\{(1,0),(1,1)\} \\ Q_2 &= \mathbb{H}_8 = \pm\{(1,0,0),(1,1,0),(1,0,1),(1,1,1)\} \end{aligned}$$

Multiplication

$(x,0)(y,0) = (xy,0)$
$(x,0)(y,1) = (yx,1)$
$(x,1)(y,0) = (xy^*,1)$
$(x,1)(y,1) = (-y^*x,0)$

Conjugation

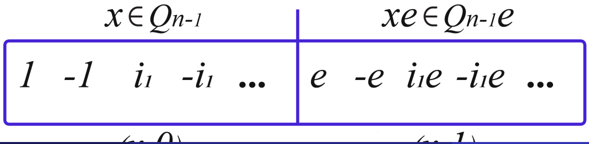
$(x,0)^* = (x^*,0)$
$(x,1)^* = (-x,1)$

Canonical generators

Definition

Let $e = i_n = (1_{Q_{n-1}}, 1) = (1, \underbrace{0, \ldots, 0}_{n-1}, 1) \in Q_n$, then $Q_n = Q_{n-1} \cup (Q_{n-1} i_n)$, and $Q_n = \langle i_1, i_2, \ldots, i_n \rangle$. We call $i_1, i_2, \ldots, i_n$ **canonical generators** of Q_n.

$$
\begin{aligned}
\textbf{Complex group (abelian) } Q_1 &= \mathbb{C}_4 = \langle \underline{i_1} \rangle = \{1, -1, i_1, -i_1\} \\
\textbf{Quaternion group (not abelian) } Q_2 &= \mathbb{H}_8 = \langle i_1, i_2 \rangle = \pm\{1, i_1, i_2, i_1 i_2\} \\
\textbf{Octonion loop (Moufang) } Q_3 &= \mathbb{O}_{16} = \langle i_1, i_2, i_3 \rangle = \\
&= \pm\{1, i_1, i_2, i_1 i_2, i_3, i_1 i_3, i_2 i_3, i_1 i_2 i_3\} \\
\textbf{Sedenion loop (not Moufang) } Q_4 &= \mathbb{S}_{32} = \langle i_1, i_2, i_3, i_4 \rangle
\end{aligned}
$$



Properties

Let $x, y \in Q_n$

- **Conjugate**: $x^* = -x$ for $x \neq \pm 1$, $1^* = 1$, $(-1)^* = -1$
- **Inverse**: $x^{-1} = x^*$
- **Order**: $|x| = 4$ for $x \neq \pm 1$, $|1| = 1$, $|-1| = 2$
- **Size**: $|Q_n| = 2^{n+1}$
- **Embedding**: $(Q_{n-1}, \cdot) < (Q_n, \cdot)$
- **Diassociativity** *(Culbert, 2007)*: any two elements generate a group $\langle x, y \rangle \leq \mathbb{H}_8$ (quaternion group)
- Q_n **is Hamiltonian**: any subloop S is normal in Q_n
  $xS = Sx, \quad (xS)y = x(Sy), \quad x(yS) = (xy)S$

Center, commutators and associators

Let $x, y, z \in Q_n$.

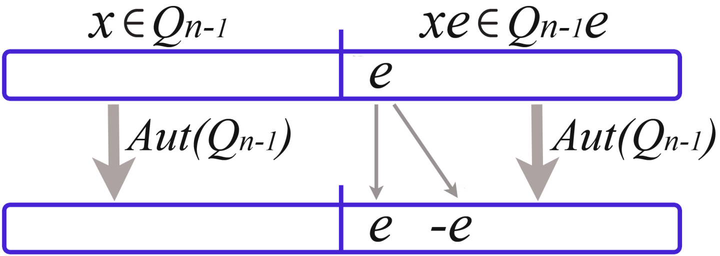
Center $Z(Q_n) = \{1, -1\}$ when $n > 1$, $Z(Q_n) = Q_n$ when $n = 1$.

Commutator $[x, y] = -1$ when $\langle x, y\rangle \cong \mathbb{H}_8$ and $[x, y] = 1$ when $\langle x, y\rangle < \mathbb{H}_8$.

Associator $[x, y, z] = 1$ or $[x, y, z] = -1$, moreover, if $\langle x, y, z\rangle \cong \mathbb{O}_{16}$, then $[x, y, z] = -1$.

Automorphism group

	Size observation	Structure
$Aut(\mathbb{C}_4)$	2	$\mathbb{Z}_2$
$Aut(\mathbb{H}_8)$	24	$\mathbb{S}_4$
$Aut(\mathbb{O}_{16})$	$1344 = 8 \cdot 168$	Extension of $(\mathbb{Z}_2)^3$ by $PSL_2(7)$
$Aut(\mathbb{S}_{32})$	$2688 = 1344 \cdot 2$	$Aut(\mathbb{O}_{16}) \times \mathbb{Z}_2$
$Aut(\mathbb{T}_{64})$	$5376 = 2688 \cdot 2$	$Aut(\mathbb{S}_{32}) \times \mathbb{Z}_2$
$\ldots$		
$Aut(Q_n)$	$1344 \cdot 2^{n-3}$	$Aut(Q_{n-1}) \times \mathbb{Z}_2$



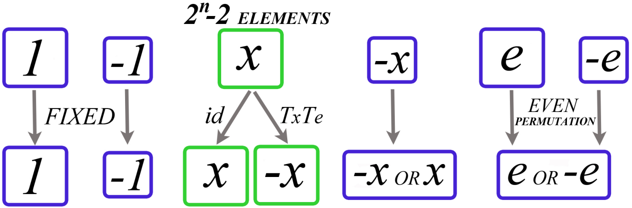
Inner mapping group

Lemma

Elements of $Mlt(Q)$ are even permutations.

Theorem

$Inn(Q)$ is an elementary abelian 2-group of order 2^{2^n-2}.
Every $f \in Inn(Q)$ is a product of disjoint transpositions of the form $(x, -x)$.
$Inn(Q) = \langle T_x T_e = (x, -x)(e, -e) \mid x \in Q, x \neq \pm 1, \pm e \rangle$.

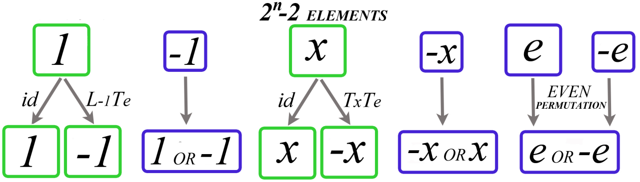


Nonassociative Cayley-Dickson loops are not automorphic, i.e., $Inn(Q) \not\leq Aut(Q)$.

Multiplication group

Theorem

$Mlt(Q) \cong (Inn(Q) \times Z(Q)) \rtimes K.$
$Mlt(Q) \cong (\mathbb{Z}_2)^{2^n-1} \rtimes (\mathbb{Z}_2)^n.$



$Inn(Q) \times Z(Q)$

Lemma

$K = \langle L_{i_m}\psi_m \mid i_m$ *canonical generator of* $Q, \psi_m \in Inn(Q)\rangle$ *such that* $|x| = 2\ \forall x \in K$. *Then* K *is an elementary abelian 2-group of size* 2^n.

Example $Q = \mathbb{O}_{16}$

$$Inn(Q) \times Z(Q) = \langle L_{-1} T_{i_3}, T_{i_1} T_{i_3}, T_{i_2} T_{i_3}, T_{i_1 i_2} T_{i_3}, T_{i_1 i_3} T_{i_3}, T_{i_2 i_3} T_{i_3}, T_{i_1 i_2 i_3} T_{i_3} \rangle$$

$$K = \langle L_{i_1} \psi_1, L_{i_2} \psi_2, L_{i_3} \psi_3 \rangle$$

$$Mlt(Q) \cong (Inn(Q) \times Z(Q)) \rtimes K$$

$$(n,k)(\tilde{n},\tilde{k}) = (n(k^{-1}\tilde{n}k), k\tilde{k}), \quad n, \tilde{n} \in Inn(Q) \times Z(Q), \quad k, \tilde{k} \in K$$

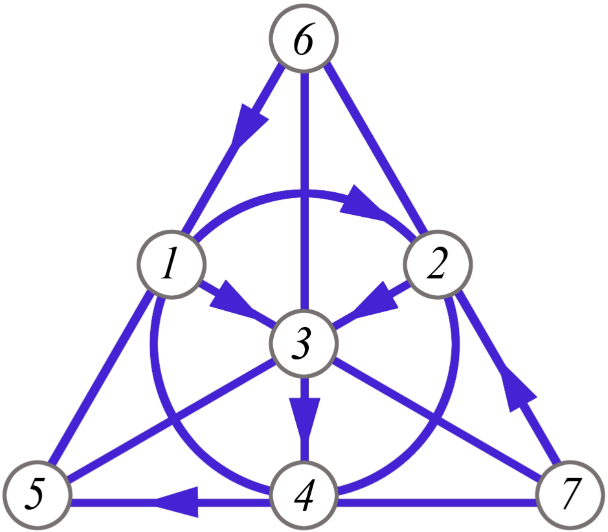
$$(k_1)^{-1} \left(\begin{array}{c} L_{-1} T_{i_3} \\ \hline T_{i_1} T_{i_3} \\ T_{i_2} T_{i_3} \\ T_{i_1 i_2} T_{i_3} \\ \hline T_{i_1 i_3} T_{i_3} \\ T_{i_2 i_3} T_{i_3} \\ T_{i_1 i_2 i_3} T_{i_3} \end{array} \right) k_1 = \left(\begin{array}{c|ccc|ccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ \hline 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{array} \right) \left(\begin{array}{c} L_{-1} T_{i_3} \\ \hline T_{i_1} T_{i_3} \\ T_{i_2} T_{i_3} \\ T_{i_1 i_2} T_{i_3} \\ \hline T_{i_1 i_3} T_{i_3} \\ T_{i_2 i_3} T_{i_3} \\ T_{i_1 i_2 i_3} T_{i_3} \end{array} \right)$$

Table: Action of $k_1 = L_{i_1} \psi_1$ on the basis of $Inn(Q) \times Z(Q)$

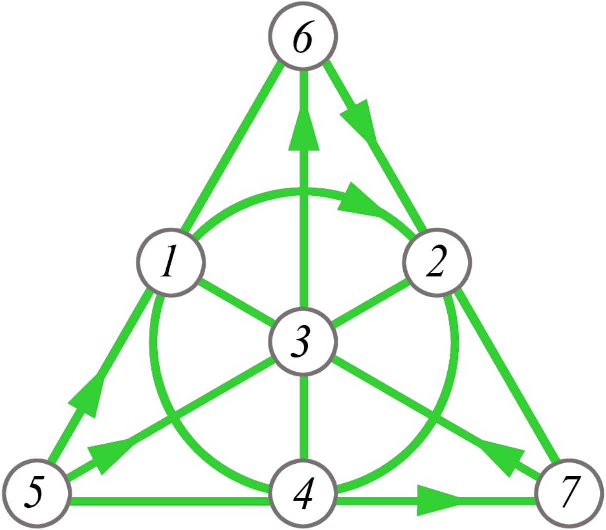
Isomorphism classes of maximal subloops

Size(Q_n)	Max subloops	Isomorphism classes	Representatives
$\mathbb{C}_4$	1	1	$\mathbb{R}_2$
$\mathbb{H}_8$	3	1	$\mathbb{C}_4$
$\mathbb{O}_{16}$	7	1	$\mathbb{H}_8$
$\mathbb{S}_{32}$	15	2	$\mathbb{O}_{16}$ and $\tilde{\mathbb{O}}_{16}$
$\mathbb{T}_{64}$	31	4	$\mathbb{S}_{32}$, $\tilde{\mathbb{S}}^1_{32}$, $\tilde{\mathbb{S}}^2_{32}$, $\tilde{\mathbb{S}}^3_{32}$
Q_{128}	63	8	
Q_{256}	127	16	

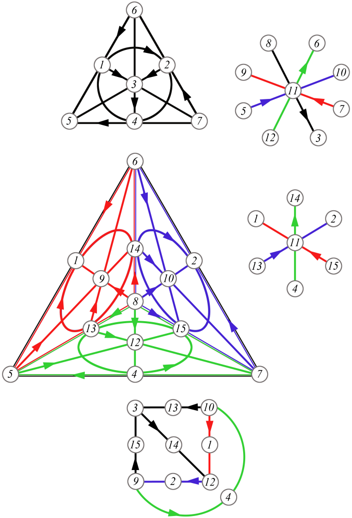
Octonion loop $\mathbb{O}_{16}$



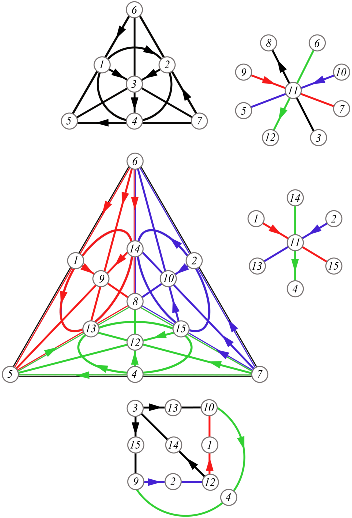
Quasioctonion loop $\tilde{\mathbb{O}}_{16}$



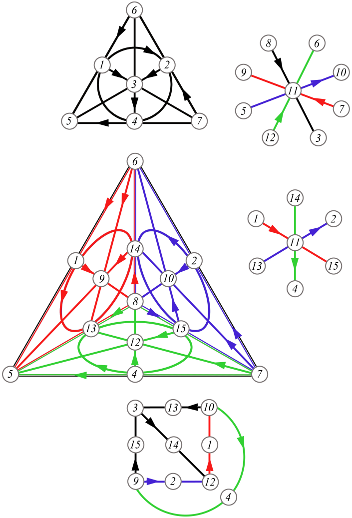
Sedenion loop $\mathbb{S}_{32}$



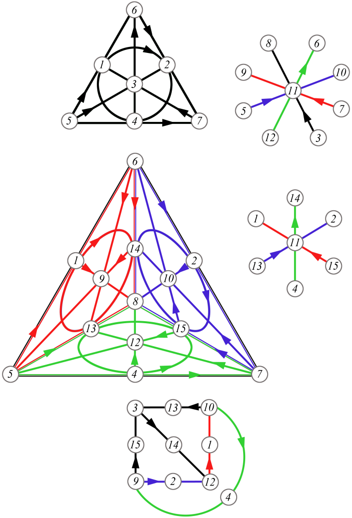
Quasisedenion loop $\tilde{\mathbb{S}}^1_{32}$



Quasisedenion loop $\tilde{\mathbb{S}}_{32}^{2}$



Quasisedenion loop $\tilde{\mathbb{S}}_{32}^{3}$



Number of subloops

Theorem

$Q_n/\{1,-1\} \cong (\mathbb{Z}_2)^n$.

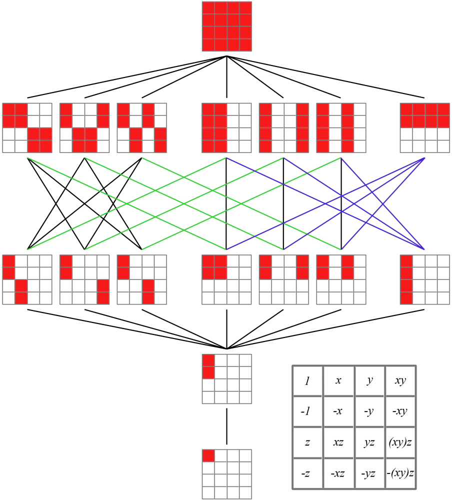
Theorem

Q_n contains 1 subloop of orders 1 and 2, and

$$\eta(k) = \prod_{j=1}^{k-1} \frac{(2^{n-j+1}-1)}{(2^{k-j}-1)} \text{ subloops of order } 2^k, \quad 2 \leq k \leq n.$$

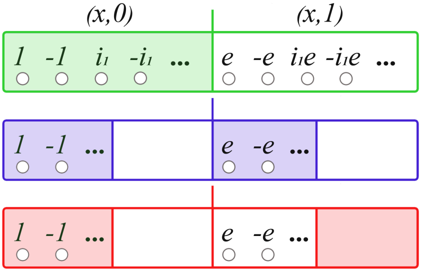
Also, $\eta(k) = \eta(n-k+2)$.

Subloop lattice of $\mathbb{O}_{16}$



1	*x*	*y*	*xy*
-1	*-x*	*-y*	*-xy*
z	*xz*	*yz*	*(xy)z*
-z	*-xz*	*-yz*	*-(xy)z*

Maximal subloops



Theorem

Let D be a subloop of Q_{n-1} of index 2. A subloop of Q_n of index 2 is either Q_{n-1},
or $B = D \cup Di_n$,
or $C = D \cup (Q_{n-1} \backslash D)\, i_n$.
$C \ncong Q_{n-1}$, and $C \ncong B$.

Associator-commutator calculus

Suppose that $cl(Q) \leq 2$ and $Z(Q)$ has exponent 2. Then

$$[xy, z][x, y][x, zy][y, z][x, y, z][z, y, x] = 1.$$

Thus in a Cayley-Dickson loop

$$[x, y, z] = [z, y, x].$$

Also, in a Cayley-Dickson loop

$$\begin{aligned} [x, xy, z] &= [x, y, z], \\ [xy, y, xz] &= [y, x, z]. \end{aligned}$$

The value of $[xy, x, z]$ is invariant under any permutation of x, y, z. In particular,

$$[xy, x, z] = [xy, y, z].$$

From Q_{n-1} to Q_n

S is a subloop of order 2^n in a Cayley-Dickson loop.
Want to **extend** it to a subloop T of order 2^{n+1} **by adjoining an element** u.
Then $T = S \cup Su$.
The **multiplication** in T **is given by**

$$\begin{aligned}
x \cdot y &= xy,\\
x \cdot yu &= [x, y, u](xy)u,\\
xu \cdot y &= [y, xu]y \cdot xu = [y, xu][y, x, u]yx \cdot u = [y, xu][x, y][y, x, u]xy \cdot u,\\
xu \cdot yu &= [y, u]xu \cdot uy = [y, u][x, u, uy]x(u \cdot uy) = -[y, u][x, u, uy]xy\\
&= (\text{note: } [x, u, uy] = [uy, u, x] = [xy, x, u]) = -[y, u][xy, x, u]xy,
\end{aligned}$$

where $x,\ y \in S$.
We only need to know the associators $[x, y, u]$ for $x,\ y \in S$.

Subloops of size 16

$|S| = 8$, $S = \pm\{1, a, b, ab\}$. The associators we need to know are:

$$\begin{aligned}
&[a, b, u],\\
&[a, ab, u] = [a, b, u],\\
&[b, a, u],\\
&[b, ab, u] = [b, ba, u] = [b, a, u],\\
&[ab, a, u],\\
&[ab, b, u] = [ab, a, u].
\end{aligned}$$

All we need to describe T are the 3 associators $[a, b, u]$, $[b, a, u]$, $[ab, b, u]$.

Subloops of size 16

1	x	y	xy	z	xz	yz	(xy)z
x	-1	xy	-y	xz	-z	[x,y,z](xy)z	-[x,y,z]yz
y	-xy	-1	x	yz	-[y,x,z](xy)z	-z	[y,x,z]xz
xy	y	-x	-1	(xy)z	[xy,x,z]yz	-[xy,x,z]xz	-z
z	-xz	-yz	-(xy)z	-1	x	y	xy
xz	z	[y,x,z](xy)z	-[xy,x,z]yz	-x	-1	[xy,x,z]xy	-[y,x,z]y
yz	-[x,y,z](xy)z	z	[xy,x,z]xz	-y	-[xy,x,z]xy	-1	[x,y,z]x
(xy)z	[x,y,z]yz	-[y,x,z]xz	z	-xy	[y,x,z]y	-[x,y,z]x	-1

Lemma

If x, y, z are elements of Q_n such that $|\langle x, y, z\rangle| = 16$, then either

$$\langle x, y, z\rangle \cong \mathbb{O}_{16} \text{ (octonion loop) or}$$
$$\langle x, y, z\rangle \cong \tilde{\mathbb{O}}_{16} \text{ (quasioctonion loop).}$$

Subloops of size 32

Conjecture

Let Q_n be a Cayley-Dickson loop. Every subloop of size 32 of Q_n is isomorphic to a maximal subloop of $\mathbb{T}_{64}$ (the sedenion loop $\mathbb{S}_{32}$, or one of the quasisedenion loops $\tilde{\mathbb{S}}^1_{32}, \tilde{\mathbb{S}}^2_{32}, \tilde{\mathbb{S}}^3_{32}$).

Subloops of size 32

$|S| = 16$, $S = \langle a, b, c \rangle$, $u \notin S$, $T = S \cup Su$.
To specify T we need $[x, y, u]$,
where x, $y \in S = \pm\{1, a, b, ab, c, ac, bc, (ab)c\}$.
We need:

$$[a, b, u],$$
$$[a, c, u],$$
$$[a, ab, u] = [a, b, u],$$
$$[a, ac, u] = [a, c, u],$$
$$[a, bc, u],$$
$$[a, abc, u] = [a, bc, u],$$
$$[b, a, u],$$
$$[b, c, u],$$
$$[b, ab, u] = [b, a, u],$$
$$[b, ac, u],$$
$$[ab, ac, u],$$
$$[ab, bc, u] = [ab, ac, u],$$
$$[ab, abc, u] = [ab, c, u],$$
$$[ac, a, u],$$
$$[ac, b, u],$$
$$[ac, c, u] = [ac, a, u],$$
$$[ac, ab, u],$$
$$[ac, bc, u] = [ac, ab, u],$$
$$[ac, abc, u] = [ac, b, u],$$
$$[bc, a, u],$$

Subloops of size 32

$[b, bc, u] = [b, c, u]$,
$[b, abc, u] = [b, ac, u]$,
$[c, a, u]$,
$[c, b, u]$,
$[c, ab, u]$,
$[c, ac, u] = [c, a, u]$,
$[c, bc, u] = [c, b, u]$,
$[c, abc, u] = [c, ab, u]$,
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$[abc, b, u]$,
$[abc, c, u]$,
$[abc, ab, u] = [abc, c, u]$,
$[abc, ac, u] = [abc, b, u]$,
$[abc, bc, u] = [abc, a, u]$.

More problems + references

Conjecture

If S is a subloop of a Cayley-Dickson loop Q_n, then S is a maximal subloop of a Cayley-Dickson loop Q_k, $k \leq n+1$.

Conjecture

There are 2^{n-3} isomorphism classes of maximal subloops of a Cayley-Dickson loop Q_n.

References

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- Jenya Kirshtein: *Multiplication groups and inner mapping groups of Cayley-Dickson loops*, to be submitted
- Jenya Kirshtein: *Maximal subloops of Cayley-Dickson loops*, in preparation

Thank you!