Are Commutants in Moufang Loops Normal?

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(((C \ast x) \ast y) \ast (y' \ast x')) \ast z = z \ast (((C \ast x) \ast y) \ast (y' \ast x'))
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So, *are* commutants in Moufang loops normal? This is an obvious “first” question. And it is has a (somewhat) dignified history. In his famous 1976 paper, S. Doro conjectured that in Moufang loops with trivial nucleus, the answer is “yes”. But Doro—who was, let’s face it, a pretty clever guy—was unable to prove it. Doro’s conjecture—recast more generally and as a question, viz, the title of this talk—has been part of the loop theory folklore since then.
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**Curious Fact:** There is no known equational proof of this (relatively) uncomplicated theorem. There should be one. And Prover9 should be able to find it.
Machinery/Notation/Terminology:

Let's work "elementwise":

$C$ is an arbitrary commutant element; $A$ and $B$ are arbitrary constants.

Set $D$ as $CR(A,B) = D$

So, the question is: is $D$ in the commutant? That is, does the following hold:

$D^*x = x^*D$
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If you can't show that $D$ is in the commutant, instead, show that $D$ has many (some?) commutant-like properties; e.g., $D^3$ is nuclear (and piles and Piles and PILES of others).
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**Theorem.** (No assumptions on $A$, $B$, or $C$). (i) $D$ commutes with cubes. (ii) If $D$ commutes with $E$, and if $D$ commutes with $F$, then $D$ commutes with $E \ast F$ (so if $L$ is generated by cubes, then $D$ is in the commutant).