ON A HOMOMORPHISM PROPERTY OF HOOPS

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Abstract. We present a syntactic proof that equation

$$e \to (b \cdot c) \;=\; (e \to b) \cdot (e \to c)$$

is satisfied in a hoop $\mathbf{A}$ for any idempotent $e \in A$ and all $b, c \in A$. The theorem both answers a question and generalizes a result of Ferreirim [6].

1. Introduction

A *hoop* is an algebra $\langle A; \cdot, \to, 1\rangle$ of type $\langle 2, 2, 0\rangle$ that satisfies the identities:

$$x \cdot (y \cdot z) \approx (x \cdot y) \cdot z \tag{M1}$$

$$x \cdot y \approx y \cdot x \tag{M2}$$

$$x \cdot \mathbf{1} \approx x \tag{M3}$$

$$x \to x \approx \mathbf{1} \tag{M4}$$

$$(x \to y) \cdot x \approx (y \to x) \cdot y \tag{M5}$$

$$(x \cdot y) \to z \approx x \to (y \to z). \tag{M6}$$

We denote the variety of hoops by HO.

Hoops were first investigated by Büchi and Owens in an unpublished manuscript [5] of 1975, and they have since been studied by Ferreirim [6, 7], Blok and Ferreirim [2, 3], Aglianó and Panti [1] and Blok and Pigozzi [4] among others. The study of hoops is motivated by their occurrence both in universal algebra and algebraic logic. Typical examples of hoops include both Brouwerian semilattices and the positive cones of lattice ordered Abelian groups, while hoops structurally enriched with normal multiplicative operators naturally generalize the normal Boolean algebras with operators of Jónsson and Tarski [8, 9]. For details, see in particular Blok and Pigozzi [4].

For a hoop $\mathbf{A}$, an *idempotent* is an element $e \in A$ with the property that $e \cdot e = e$. It is well known and easy to see [4, Lemma 1.9(v)] that for any hoop $\mathbf{A}$ and idempotent $e \in A$, the following equation is satisfied for all $b, c \in A$:

$$e \to (b \to c) = (e \to b) \to (e \to c). \tag{1}$$

For any $\langle \to, \cdot, \mathbf{1} \rangle$-term $t := t(\vec{x})$, the *k-th iterated power* t^k, $0 < k < \omega$, is defined recursively by:

$$t^0 := \mathbf{1}$$
$$t^k := t \cdot t^{k-1}.$$

A hoop is said to be *k-potent* if it satisfies the identity:

$$x^k \approx x^{k-1}.$$

In her Ph.D. thesis [6, Chapter 3, Lemma 1.10], Ferreirim shows that in addition to (1), any k-potent hoop $\mathbf{A}$ satisfies the following equation for any idempotent $e \in A$ and all $b, c \in A$:

$$e \to (b \cdot c) = (e \to b) \cdot (e \to c). \tag{2}$$

Together, (1) and (2) assert that the map $a \mapsto (e \to a)$ is an endomorphism for any k-potent hoop $\mathbf{A}$ and fixed idempotent $e \in A$. This implies in particular that $\mathsf{H}(\mathbf{A}) \subseteq \mathsf{IS}(\mathbf{A})$ for any finite hoop $\mathbf{A}$ [6, Chapter 3, Lemma 1.11]. (Here $\mathsf{H}(\mathbf{A})$ and $\mathsf{IS}(\mathbf{A})$ denote the class of all homomorphic images of $\mathbf{A}$ and the class of all isomorphic copies of subalgebras of $\mathbf{A}$, respectively.) This strong property plays an important role in the proofs of several results in the theory of hoops, including Ferreirim's characterization of finitely based varieties of k-potent hoops [6, Chapter 3, Theorem 1.13], and her characterization of varieties of hoops in which every subquasivariety is itself a variety [6, Chapter 3, Theorem 2.13].

The proof of (2) given by Ferreirim in [6, Chapter 3, Lemma 1.10] relies on a sophisticated model-theoretic argument that exploits her characterization of the subdirectly irreducible k-potent hoops [6, Chapter 2, Theorem 3.12]. Immediately following her proof of (2) Ferreirim remarks [6, p. 58]: 'A syntactic proof of statement (2) $(e \to b) \cdot (e \to c) = e \to (b \cdot c)$ in Lemma 1.10 would certainly be more elegant. We couldn't find one and propose it as an open problem.'

In this note, we present a solution to Ferreirim's problem by exhibiting a syntactic proof of her equation (2). Moreover, our solution generalizes Ferreirim's result to all hoops, since our proof does not assume k-potency. In particular, our proof holds for the subvariety L of *Łukasiewicz hoops*, namely the class of all hoops satisfying the commutative identity $(x \to y) \to y \approx (y \to x) \to x$; the significance of L in the theory of HO lies in a result due to Ferreirim [6, Chapter 3, Corollary 3.4] that shows Łukasiewicz hoops are, in a precise technical sense, the *building blocks* of arbitrary hoops. We note that Ferreirim's proof of (2) does not extend to L, since the variety of Łukasiewicz hoops is not k-potent for any $k < \omega$ by [6, Chapter 2, Corollary 4.17] and [4, Corollary 5.5].

2. The Proof

In the following (machine-oriented) proof, the justification $[i \to j]$ indicates paramodulation from i into j, that is, unifying the left-hand side of i with a subterm of j, instantiating j with the corresponding substitution, and replacing the subterm with the corresponding instance of the right-hand side of i.

1. $x \cdot (y \cdot z) \ = \ (x \cdot y) \cdot z$ [M1]
2. $x \cdot y \ = \ y \cdot x$ [M2]
3. $x \cdot 1 \ = \ x$ [M3]
4. $x \to x \ = \ 1$ [M4]
5. $(x \to y) \cdot x \ = \ (y \to x) \cdot y$ [M5]
6. $(x \cdot y) \to z \ = \ x \to (y \to z)$ [M6]
7. $e \cdot e \ = \ e$ [e is an idempotent]
8. $x \cdot (y \cdot z) \ = \ z \cdot (x \cdot y)$ $[1 \to 2]$
9. $(x \cdot y) \cdot z \ = \ (y \cdot z) \cdot x$ $[1 \to 2]$
10. $(x \cdot y) \cdot z \ = \ x \cdot (z \cdot y)$ $[2 \to 1]$
11. $1 \cdot x \ = \ x$ $[2 \to 3]$
12. $x \to x \ = \ y \to y$ $[4 \to 4]$
13. $x \cdot (y \to y) \ = \ x$ $[4 \to 3]$
14. $(x \to y) \cdot x \ = \ y \cdot (y \to x)$ $[2 \to 5]$
15. $((x \to y) \cdot x) \to z \ = \ (y \to x) \to (y \to z)$ $[5 \to 6]$
16. $(x \cdot y) \to z \ = \ y \to (x \to z)$ $[2 \to 6]$
17. $e \to (e \to x) \ = \ e \to x$ $[7 \to 6]$
18. $e \cdot (e \cdot x) \ = \ e \cdot x$ $[7 \to 1]$
19. $((x \to y) \cdot x) \to y \ = \ z \to z$ $[6 \to 12]$

20. $(x \cdot y) \to y \;=\; x \to (z \to z)$ [$12 \to 6$]
21. $x \cdot (y \to (z \to (y \cdot z))) \;=\; x$ [$6 \to 13$]
22. $(x \cdot y) \cdot z \;=\; y \cdot (z \cdot x)$ [$2 \to 8$]
23. $x \cdot (x \to y) \;=\; y \cdot (y \to x)$ [$2 \to 14$]
24. $(x \cdot (x \to y)) \to z \;=\; (y \to x) \to (y \to z)$ [$14 \to 6$]
25. $e \cdot ((x \to e) \cdot x) \;=\; e \cdot (e \to x)$ [$14 \to 18$]
26. $(x \cdot (x \to y)) \to z \;=\; y \to ((y \to x) \to z)$ [$14 \to 16$]
27. $x \to (y \to z) \;=\; y \to (x \to z)$ [$6 \to 16$]
28. $x \cdot (y \to (z \to (z \cdot y))) \;=\; x$ [$16 \to 13$]
29. $((x \to y) \cdot x) \to x \;=\; z \to z$ [$5 \to 19$]
30. $x \to ((x \to y) \to y) \;=\; z \to z$ [$16 \to 19$]
31. $x \to (y \to y) \;=\; z \to (x \to z)$ [$20 \to 16$]
32. $(x \to (y \cdot x)) \cdot ((x \to (y \cdot x)) \to y) \;=\; y$ [$21 \to 23$]
33. $e \to (x \to (e \to y)) \;=\; x \to (e \to y)$ [$17 \to 27$]
34. $(x \to y) \cdot ((x \to y) \to z) \;=\; z \cdot (x \to (z \to y))$ [$27 \to 23$]
35. $x \cdot (y \to ((y \to z) \to z)) \;=\; x$ [$27 \to 13$]
36. $(x \to y) \to (z \to z) \;=\; u \to u$ [$20 \to 29$]
37. $((x \to y) \to y) \cdot (((x \to y) \to y) \to x) \;=\; x$ [$23 \to 35$]
38. $x \cdot (y \to (z \to z)) \;=\; x$ [$36 \to 35$]
39. $x \cdot (y \to (z \to y)) \;=\; x$ [$31 \to 38$]
40. $x \cdot ((y \cdot z) \to z) \;=\; x$ [$20 \to 38$]
41. $x \cdot ((y \cdot z) \to y) \;=\; x$ [$16 \to 38$]
42. $((x \to y) \to y) \cdot (x \to y) \;=\; y$ [$14 \to 39$]
43. $x \to (y \to x) \;=\; 1$ [$11 \to 39$]
44. $(x \cdot y) \to x \;=\; 1$ [$6 \to 43$]
45. $(x \cdot (y \cdot z)) \to (z \cdot x) \;=\; 1$ [$22 \to 44$]
46. $x \cdot (((y \cdot z) \cdot u) \to y) \;=\; x$ [$9 \to 40$]
47. $x \cdot (x \to (y \cdot x)) \;=\; y \cdot x$ [$23 \to 40$]
48. $x \cdot (x \to (x \cdot y)) \;=\; x \cdot y$ [$23 \to 41$]
49. $(x \to x) \to x \;=\; x$ [$13 \to 42$]
50. $((x \to y) \to y) \to ((x \to y) \to z) \;=\; y \to z$ [$42 \to 6$]
51. $(x \to ((x \to y) \to y)) \to z \;=\; z$ [$30 \to 49$]
52. $(x \cdot y) \to ((z \to y) \cdot x) \;=\; 1$ [$42 \to 45$]
53. $(x \to y) \to (x \to z) \;=\; (y \to x) \to (y \to z)$ [$6 \to 15$]
54. $x \cdot ((y \cdot z) \to (u \to y)) \;=\; x$ [$6 \to 46$]
55. $x \cdot (x \to (y \cdot x)) \;=\; x \cdot y$ [$2 \to 47$]
56. $x \to ((x \to (x \cdot y)) \to z) \;=\; (x \cdot y) \to z$ [$48 \to 6$]
57. $e \to ((x \to e) \cdot e) \;=\; 1$ [$7 \to 52$]
58. $e \cdot (x \to e) \;=\; e \cdot 1$ [$57 \to 55$]

59. $e \cdot (x \to e) \ = \ e$ [$3 \to 58$]
60. $e \cdot ((x \to e) \cdot y) \ = \ e \cdot y$ [$59 \to 1$]
61. $e \cdot (e \to x) \ = \ e \cdot x$ [$60 \to 25$]
62. $(e \to x) \to (e \to y) \ = \ (e \cdot x) \to y$ [$61 \to 16$]
63. $(e \to x) \to (e \to y) \ = \ x \to (e \to y)$ [$16 \to 62$]
64. $(e \to x) \to (e \to y) \ = \ e \to (x \to y)$ [$6 \to 62$]
65. $(e \to x) \cdot ((e \to x) \to (e \to y))$
$= \ (e \to y) \cdot (y \to (e \to x))$ [$63 \to 23$]
66. $x \cdot ((y \to (z \cdot u)) \to (y \to z)) \ = \ x$ [$24 \to 46$]
67. $(x \to y) \cdot ((x \to y) \to (x \to (y \cdot z))) \ = \ x \to (y \cdot z)$ [$23 \to 66$]
68. $(x \to (x \cdot y)) \to (((x \to (x \cdot y)) \to y) \to z) \ = \ y \to z$ [$28 \to 26$]
69. $(x \to (y \cdot x)) \cdot ((x \to (x \cdot y)) \to y) \ = \ y$ [$2 \to 32$]
70. $(x \to ((y \to z) \cdot x)) \cdot (y \to ((x \to ((y \to z) \cdot x)) \to z))$
$= \ y \to z$ [$27 \to 32$]
71. $(x \to y) \to ((z \to x) \to (z \to y)) \ = \ 1$ [$53 \to 43$]
72. $(((x \to y) \to y) \to z) \to (x \to z) \ = \ 1$ [$51 \to 71$]
73. $(((x \to y) \to y) \to y) \cdot 1 \ = \ x \to y$ [$72 \to 37$]
74. $((x \to y) \to y) \to y \ = \ x \to y$ [$3 \to 73$]
75. $x \to ((x \to (x \cdot y)) \to z) \ = \ x \to (y \to z)$ [$6 \to 56$]
76. $(x \to y) \cdot (x \to ((x \to y) \to (y \cdot z))) \ = \ x \to (y \cdot z)$ [$66 \to 34$]
77. $(x \to y) \cdot ((((x \to y) \to y) \to y) \to (((x \to y) \to y) \to (y \cdot z)))$
$= \ ((x \to y) \to y) \to (y \cdot z)$ [$74 \to 67$]
78. $((x \to (x \cdot y)) \to y) \cdot (y \to (y \cdot z))$
$= \ (x \to (x \cdot y)) \to (y \cdot z)$ [$68 \to 76$]
79. $(e \to x) \cdot (e \to (x \to y)) \ = \ (e \to y) \cdot (y \to (e \to x))$ [$64 \to 65$]
80. $(e \to (x \cdot e)) \cdot ((x \cdot e) \to (e \to x)) \ = \ e \to x$ [$28 \to 79$]
81. $e \to (x \cdot e) \ = \ e \to x$ [$54 \to 80$]
82. $e \to (x \cdot (e \cdot y)) \ = \ e \to (y \cdot x)$ [$22 \to 81$]
83. $e \to (x \cdot (e \cdot y)) \ = \ e \to (x \cdot y)$ [$10 \to 81$]
84. $e \to (x \cdot (e \cdot y)) \ = \ e \to (x \cdot (e \to y))$ [$61 \to 83$]
85. $e \to (x \cdot (e \to y)) \ = \ e \to (y \cdot x)$ [$82 \to 84$]
86. $e \to ((e \to x) \cdot y) \ = \ e \to (x \cdot y)$ [$2 \to 85$]
87. $(e \to x) \to (e \to (y \cdot z)) \ = \ e \to (x \to ((e \to y) \cdot z))$ [$86 \to 64$]
88. $e \to (x \to ((e \to y) \cdot z)) \ = \ e \to (x \to (y \cdot z))$ [$64 \to 87$]

89. $(((x \to (x \cdot y)) \to y) \to y)$
$\cdot (x \to ((((x \to (x \cdot y)) \to y) \to ((x \to (y \cdot x))$
$\cdot ((x \to (x \cdot y)) \to y))) \to (y \cdot x)))$
$= x \to (y \cdot x)$ [69 → 70]
90. $(x \to y) \cdot (y \to (y \cdot z)) = ((x \to y) \to y) \to (y \cdot z)$ [50 → 77]
91. $(((x \to (x \cdot y)) \to y) \to y) \to (y \cdot z)$
$= (x \to (x \cdot y)) \to (y \cdot z)$ [78 → 90]
92. $(((x \to (x \cdot y)) \to y) \to y)$
$\cdot (x \to ((((x \to (x \cdot y)) \to y) \to y) \to (y \cdot x)))$
$= x \to (y \cdot x)$ [69 → 89]
93. $(((x \to (x \cdot y)) \to y) \to y) \cdot (x \to ((x \to (x \cdot y)) \to (y \cdot x)))$
$= x \to (y \cdot x)$ [91 → 92]
94. $(((x \to (x \cdot y)) \to y) \to y) \cdot (x \to (y \to (y \cdot x)))$
$= x \to (y \cdot x)$ [75 → 93]
95. $((x \to (x \cdot y)) \to y) \to y = x \to (y \cdot x)$ [28 → 94]
96. $((x \to (y \cdot x)) \to y) \to y = x \to (y \cdot x)$ [2 → 95]
97. $e \to (((x \to ((e \to y) \cdot x)) \to (e \to y)) \to (e \to y))$
$= x \to ((e \to y) \cdot x)$ [33 → 96]
98. $e \to (x \to ((e \to y) \cdot x)) = x \to ((e \to y) \cdot x)$ [96 → 97]
99. $x \to ((e \to y) \cdot x) = e \to (x \to (y \cdot x))$ [98 → 88]
100. $x \to ((e \to y) \cdot x) = x \to (e \to (y \cdot x))$ [27 → 99]
101. $x \cdot (x \to (e \to (y \cdot x))) = (e \to y) \cdot x$ [100 → 47]
102. $(e \to x) \cdot ((e \to x) \to (e \to (x \cdot y)))$
$= (e \to y) \cdot (e \to x)$ [85 → 101]
103. $(e \to x) \cdot (e \to y) = e \to (y \cdot x)$ [67 → 102]
104. $(e \to x) \cdot (e \to y) = e \to (x \cdot y)$ [2 → 103]

This completes the proof.

Note. This proof was obtained with the assistance of the automated reasoning program Otter [10], using the method of proof sketches [12]. See [11] for examples of the application of automated reasoning to a wide range of problems in equational logic.

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Acknowledgements

This paper was written while the second author was a visitor at the School of Informatics and Engineering at the Flinders University of South Australia. The facilities and assistance provided by the University and the School are gratefully acknowledged.

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