Discrete Random Variables

Let X be a *discrete random variable* with outcomes, $x_1, x_2, \ldots, x_n$. The probability that the outcome of experiment X is x_i is $P(X = x_i)$ or $p_X(x_i)$:

- $\forall_i \;\; p_X(x_i) \geq 0$
- $\sum_{i=1}^{n} p_X(x_i) = 1$

p_X is termed the *probability mass function*.

Expected Value[1]

Let X be a discrete random variable with numerical outcomes, $\{x_1,\ldots,x_n\}$. The *expected value* of X, is defined as follows:

$$\langle X\rangle = \sum_{i=1}^{n} p_X(x_i)x_i$$

Variance

The *variance* of X is defined as the expected value of the squared difference of X and $\langle X\rangle$:

$$\left\langle [X-\langle X\rangle]^2\right\rangle = \sum_{i=1}^{n} p_X(x_i)\,[x_i-\langle X\rangle]^2$$

[1]"God is or He is not...Let us weight the gain and the loss in choosing...'God is.' If you gain, you gain all, if you lose, you lose nothing. Wager, then, unhesitatingly, that He is." – Blaise Pascal

Continuous Random Variables

The probability that a *continuous random variable*, X, has a value between a and b is computed by integrating its *probability density function (p.d.f.)* over the interval $[a,b]$:

$$P(a \leq X \leq b) = \int_a^b f_X(x)dx.$$

A p.d.f. must integrate to one:

$$\int_{-\infty}^{\infty} f_X(x)dx = 1.$$

Continuous Random Variables (contd.)

The probability that the continuous random variable, X, has any exact value, a, is 0:

$$\begin{aligned} P(X=a) &= \lim_{\Delta x \to 0} P(a \leq X \leq a + \Delta x) \\ &= \lim_{\Delta x \to 0} \int_a^{a+\Delta x} f_X(x) dx \\ &= 0. \end{aligned}$$

In general

$$P(X=a) \neq f_X(a).$$

Probability Density

The probability density at a multiplied by ε approximately equals the probability mass contained within an interval of ε width centered on a:

$$
\begin{aligned}
\varepsilon f_X(a) &\approx \int_{a-\varepsilon/2}^{a+\varepsilon/2} f_X(x)dx \\
&\approx P(a-\varepsilon/2 \leq X \leq a+\varepsilon/2)
\end{aligned}
$$

<u>Cumulative Distribution Function</u>

A continuous random variable, X, can also be defined by its *cumulative distribution function (c.d.f.)*:

$$F_X(a) = P(X \leq a) = \int_{-\infty}^{a} f_X(x)dx.$$

For any c.d.f., $F_X(-\infty) = 0$ and $F_X(\infty) = 1$. The probability that a continuous random variable, X, has a value between a and b is easily computed using the c.d.f.:

$$\begin{aligned} P(a \leq X \leq b) &= \int_{a}^{b} f_X(x)dx \\ &= \int_{-\infty}^{b} f_X(x)dx - \int_{-\infty}^{a} f_X(x)dx \\ &= F_X(b) - F_X(a). \end{aligned}$$

Cumulative Distribution Function (contd.)

The p.d.f., $f_X(x)$, can be derived from the c.d.f., $F_X(x)$:

$$\begin{aligned} f_X(x) &= \frac{d}{dx}\int_{-\infty}^{x} f_X(s)ds \\ &= \frac{dF_X(x)}{dx}. \end{aligned}$$

Expected Value

Let X be a continuous random variable. The *expected value* of X, is defined as follows:

$$\langle X \rangle = \mu = \int_{-\infty}^{\infty} x f_X(x) dx$$

Variance

The *variance* of X is defined as the expected value of the squared difference of X and $\langle X \rangle$:

$$\left\langle [X - \langle X \rangle]^2 \right\rangle = \sigma^2 = \int_{-\infty}^{\infty} [x - \langle X \rangle]^2 f_X(x) dx$$