Frames vs. Bases

- A set of vectors form a *basis* for \mathbb{R}^M if they span \mathbb{R}^M and are linearly independent.
- A set of $N \ge M$ vectors form a *frame* for \mathbb{R}^M if they span \mathbb{R}^M .



Figure 1: (A) A basis. (B) A frame with overcompleteness 3. (C) A basis. (D) A frame with overcompletness 3/2.

Frame Operator

Let \mathcal{F} consist of the *N* frame vectors, $\mathbf{f}_1 \dots \mathbf{f}_N \in \mathbb{R}^M$, where $N \ge M$. Let $\mathbf{x} \in \mathbb{R}^N$ be a representation of $\mathbf{y} \in \mathbb{R}^M$ in \mathcal{F} . It follows that

$$\mathbf{y} = x_1 \mathbf{f}_1 + x_2 \mathbf{f}_2 + \dots + x_N \mathbf{f}_N.$$

This is just the matrix vector product

$$\mathbf{y} = \mathbf{F}\mathbf{x}$$

where the *frame operator*, **F**, is the $M \times N$ matrix,

 $\mathbf{F} = \begin{bmatrix} \mathbf{f}_1 \mid \mathbf{f}_2 \mid \dots \mid \mathbf{f}_N \end{bmatrix}.$

Inverse Frame Operator (contd.)

We might guess that

$$\mathbf{x} = \mathbf{F}^{-1}\mathbf{y}$$

where $\mathbf{F}\mathbf{F}^{-1} = \mathbf{I}$. Unfortunately, because **F** is not square, it has no simple inverse. However, it has an infinite number of *right-inverses*. Each of the **x** produced when **y** is multiplied by a distinct right-inverse is a distinct representation of the vector **y** in the frame, \mathcal{F} .

Pseudoinverse

If there are an infinite number of rightinverses, then we should be able to find at least one of them. We observe that the *pseudoinverse*

$$\mathbf{F}^{+} = \mathbf{F}^{\mathrm{T}} \left(\mathbf{F} \mathbf{F}^{\mathrm{T}} \right)^{-1}$$

is a right-inverse of **F**. We call the $N \times M$ matrix, **F**⁺, an *inverse frame operator* because it maps vectors, $\mathbf{y} \in \mathbb{R}^{M}$, into representations, $\mathbf{x} \in \mathbb{R}^{N}$.

Frame Bounds

Let \mathcal{F} consist of the *N* frame vectors, $\mathbf{f}_1 \dots \mathbf{f}_N \in \mathbb{R}^M$, where $N \ge M$, and let \mathbf{F}^+ be the inverse frame operator. \mathcal{F} is a frame iff for all $\mathbf{y} \in \mathbb{R}^M$ there exist *A* and *B* where $0 < A \le B < \infty$ and where

$$rac{1}{B}||\mathbf{y}||^2 \leq ||\mathbf{F}^+\mathbf{y}||^2 \leq rac{1}{A}||\mathbf{y}||^2.$$

A and B are called the *frame bounds*.

Dual Frame

If \mathcal{F} consists of the *N* frame vectors, $\mathbf{f}_1 \dots \mathbf{f}_N \in \mathbb{R}^M$, with inverse frame operator \mathbf{F}^+ , then the *dual frame*, $\widetilde{\mathcal{F}}$, consists of the *N* frame vectors, $\widetilde{\mathbf{f}}_1 \dots \widetilde{\mathbf{f}}_N \in \mathbb{R}^M$:

$$\left(\mathbf{F}^{+}\right)^{\mathrm{T}} = \left[\widetilde{\mathbf{f}}_{1} \mid \widetilde{\mathbf{f}}_{2} \mid \ldots \mid \widetilde{\mathbf{f}}_{N}\right].$$

Let $\widetilde{\mathbf{x}} \in \mathbb{R}^N$ be a representation of $\mathbf{y} \in \mathbb{R}^M$ in $\widetilde{\mathcal{F}}$. It follows that

$$\mathbf{y} = \left(\mathbf{F}^+\right)^{\mathrm{T}} \widetilde{\mathbf{x}}.$$

Consequently, $(\mathbf{F}^+)^{\mathrm{T}}$ is the frame operator for the dual frame, $\widetilde{\mathcal{F}}$.

Dual Frame (contd.)

Because \mathbf{F}^{T} is a right inverse of $(\mathbf{F}^{+})^{\mathrm{T}}$: $(\mathbf{F}^{+})^{\mathrm{T}} \mathbf{F}^{\mathrm{T}} = \mathbf{I}.$ It follows that \mathbf{F}^{T} is the inverse frame operator for the dual frame, $\widetilde{\mathcal{F}}$, and $A||\mathbf{y}||^{2} \leq ||\mathbf{F}^{\mathrm{T}}\mathbf{y}||^{2} \leq B||\mathbf{y}||^{2}.$

for all $\mathbf{y} \in \mathbb{R}^M$.

Tight-Frames

If A = B then

$$||\mathbf{F}^{\mathrm{T}}\mathbf{y}||^{2} = A||\mathbf{y}||^{2}$$

and \mathcal{F} is said to be a *tight-frame*. When \mathcal{F} is a tight-frame,

$$\mathbf{F}^+ = \frac{1}{A} \mathbf{F}^{\mathrm{T}}.$$

If $||\mathbf{f}_i|| = 1$ for all frame vectors, \mathbf{f}_i , then *A* equals the overcompleteness of the representation. When A = B = 1, then \mathcal{F} is an *orthonormal basis* and $\mathcal{F} = \widetilde{\mathcal{F}}$.

Summary of Notation

- $\mathbf{y} \in \mathbb{R}^M$ a vector.
- $\mathbf{x} \in \mathbb{R}^N$ a representation of \mathbf{y} in \mathcal{F} .
- $\mathbf{f}_1 \dots \mathbf{f}_N \in \mathbb{R}^M$ where $N \ge M$ frame vectors for \mathcal{F} .
- $\mathbf{F} = \begin{bmatrix} \mathbf{f}_1 & \mathbf{f}_2 & \dots & \mathbf{f}_N \end{bmatrix}$ frame operator for \mathcal{F} .
- $\mathbf{F}: \mathbb{R}^N \to \mathbb{R}^M$.
- $\mathbf{F}^+ = \mathbf{F}^T (\mathbf{F}^T \mathbf{F})^{-1}$ inverse frame operator for \mathcal{F} .
- $\mathbf{F}^+ : \mathbb{R}^M \to \mathbb{R}^N$.
- $0 < A \leq B < \infty$ bounds for \mathcal{F} .

Summary of Notation (contd.)

- $\widetilde{\mathbf{x}} \in \mathbb{R}^M$ a representation of \mathbf{y} in $\widetilde{\mathcal{F}}$.
- $\widetilde{\mathbf{f}}_1 \dots \widetilde{\mathbf{f}}_N \in \mathbb{R}^M$ frame vectors for $\widetilde{\mathcal{F}}$.
- $(\mathbf{F}^+)^{\mathrm{T}} = \left[\widetilde{\mathbf{f}}_1 \mid \widetilde{\mathbf{f}}_2 \mid \dots \mid \widetilde{\mathbf{f}}_N \right] \text{frame}$ operator for $\widetilde{\mathcal{F}}$.

•
$$(\mathbf{F}^+)^{\mathrm{T}} : \mathbb{R}^N \to \mathbb{R}^M.$$

• \mathbf{F}^{T} – inverse frame operator for $\widetilde{\mathcal{F}}$.

•
$$\mathbf{F}^{\mathrm{T}}: \mathbb{R}^{M} \to \mathbb{R}^{N}.$$

• $0 < \frac{1}{B} \le \frac{1}{A} < \infty$ – bounds for \mathcal{F} .