Frames vs. Bases

- A set of vectors form a basis for $\mathbb{R}^{M}$ if they span $\mathbb{R}^{M}$ and are linearly independent.
- A set of $N \geq M$ vectors form a frame for $\mathbb{R}^{M}$ if they span $\mathbb{R}^{M}$.


Figure 1: (A) A basis. (B) A frame with overcompleteness 3. (C) A basis. (D) A frame with overcompletness $3 / 2$.

## Frame Operator

Let $\mathcal{F}$ consist of the $N$ frame vectors, $\mathbf{f}_{1} \ldots \mathbf{f}_{N} \in \mathbb{R}^{M}$, where $N \geq M$. Let $\mathbf{x} \in$ $\mathbb{R}^{N}$ be a representation of $\mathbf{y} \in \mathbb{R}^{M}$ in $\mathcal{F}$. It follows that

$$
\mathbf{y}=x_{1} \mathbf{f}_{1}+x_{2} \mathbf{f}_{2}+\cdots+x_{N} \mathbf{f}_{N}
$$

This is just the matrix vector product

$$
\mathbf{y}=\mathbf{F} \mathbf{x}
$$

where the frame operator, $\mathbf{F}$, is the $M \times$ $N$ matrix,

$$
\mathbf{F}=\left[\mathbf{f}_{1}\left|\mathbf{f}_{2}\right| \ldots \mid \mathbf{f}_{N}\right] .
$$

## Inverse Frame Operator (contd.)

We might guess that

$$
\mathbf{x}=\mathbf{F}^{-1} \mathbf{y}
$$

where $\mathbf{F F}^{-1}=\mathbf{I}$. Unfortunately, because $\mathbf{F}$ is not square, it has no simple inverse. However, it has an infinite number of right-inverses. Each of the $\mathbf{x}$ produced when $\mathbf{y}$ is multiplied by a distinct rightinverse is a distinct representation of the vector $\mathbf{y}$ in the frame, $\mathcal{F}$.

Pseudoinverse
If there are an infinite number of rightinverses, then we should be able to find at least one of them. We observe that the pseudoinverse

$$
\mathbf{F}^{+}=\mathbf{F}^{\mathrm{T}}\left(\mathbf{F} \mathbf{F}^{\mathrm{T}}\right)^{-1}
$$

is a right-inverse of $\mathbf{F}$. We call the $N \times$ $M$ matrix, $\mathbf{F}^{+}$, an inverse frame operator because it maps vectors, $\mathbf{y} \in \mathbb{R}^{M}$, into representations, $\mathbf{x} \in \mathbb{R}^{N}$.

## Frame Bounds

Let $\mathcal{F}$ consist of the $N$ frame vectors, $\mathbf{f}_{1} \ldots \mathbf{f}_{N} \in \mathbb{R}^{M}$, where $N \geq M$, and let $\mathbf{F}^{+}$ be the inverse frame operator. $\mathcal{F}$ is a frame iff for all $\mathbf{y} \in \mathbb{R}^{M}$ there exist $A$ and $B$ where $0<A \leq B<\infty$ and where

$$
\frac{1}{B}\|\mathbf{y}\|^{2} \leq\left\|\mathbf{F}^{+} \mathbf{y}\right\|^{2} \leq \frac{1}{A}\|\mathbf{y}\|^{2} .
$$

$A$ and $B$ are called the frame bounds.

## Dual Frame

If $\mathcal{F}$ consists of the $N$ frame vectors, $\mathbf{f}_{1} \ldots \mathbf{f}_{N} \in \mathbb{R}^{M}$, with inverse frame operator $\mathbf{F}^{+}$, then the dual frame,$\widetilde{\mathcal{F}}$, consists of the $N$ frame vectors, $\widetilde{\mathbf{f}}_{1} \ldots \widetilde{\mathbf{f}}_{N} \in$ $\mathbb{R}^{M}$ :

$$
\left(\mathbf{F}^{+}\right)^{\mathrm{T}}=\left[\widetilde{\mathbf{f}}_{1}\left|\widetilde{\mathbf{f}}_{2}\right| \ldots \mid \widetilde{\mathbf{f}}_{N}\right] .
$$

Let $\widetilde{\mathbf{x}} \in \mathbb{R}^{N}$ be a representation of $\mathbf{y} \in$ $\mathbb{R}^{M}$ in $\widetilde{\mathcal{F}}$. It follows that

$$
\mathbf{y}=\left(\mathbf{F}^{+}\right)^{\mathrm{T}} \widetilde{\mathbf{x}}
$$

Consequently, $\left(\mathbf{F}^{+}\right)^{\mathrm{T}}$ is the frame operator for the dual frame, $\mathcal{F}$.

## Dual Frame (contd.)

Because $\mathbf{F}^{\mathrm{T}}$ is a right inverse of $\left(\mathbf{F}^{+}\right)^{\mathrm{T}}$ :

$$
\left(\mathbf{F}^{+}\right)^{\mathrm{T}} \mathbf{F}^{\mathrm{T}}=\mathbf{I} .
$$

It follows that $\mathbf{F}^{\mathrm{T}}$ is the inverse frame operator for the dual frame, $\mathcal{F}$, and

$$
A\|\mathbf{y}\|^{2} \leq\left\|\mathbf{F}^{\mathrm{T}} \mathbf{y}\right\|^{2} \leq B\|\mathbf{y}\|^{2}
$$

for all $\mathbf{y} \in \mathbb{R}^{M}$.

## Tight-Frames

If $A=B$ then

$$
\left\|\mathbf{F}^{\mathrm{T}} \mathbf{y}\right\|^{2}=A\|\mathbf{y}\|^{2}
$$

and $\mathcal{F}$ is said to be a tight-frame. When $\mathcal{F}$ is a tight-frame,

$$
\mathbf{F}^{+}=\frac{1}{A} \mathbf{F}^{\mathrm{T}} .
$$

If $\left\|\mathbf{f}_{i}\right\|=1$ for all frame vectors, $\mathbf{f}_{i}$, then $A$ equals the overcompleteness of the representation. When $A=B=1$, then $\mathcal{F}$ is an orthonormal basis and $\mathcal{F}=\mathcal{F}$.

Summary of Notation

- $\mathbf{y} \in \mathbb{R}^{M}$ - a vector.
- $\mathbf{x} \in \mathbb{R}^{N}$ - a representation of $\mathbf{y}$ in $\mathcal{F}$.
- $\mathbf{f}_{1} \ldots \mathbf{f}_{N} \in \mathbb{R}^{M}$ where $N \geq M$ - frame vectors for $\mathcal{F}$.
$\bullet \mathbf{F}=\left[\mathbf{f}_{1}\left|\mathbf{f}_{2}\right| \ldots \mid \mathbf{f}_{N}\right]$ - frame operator for $\mathcal{F}$.
- $\mathbf{F}: \mathbb{R}^{N} \rightarrow \mathbb{R}^{M}$.
- $\mathbf{F}^{+}=\mathbf{F}^{\mathrm{T}}\left(\mathbf{F}^{\mathrm{T}} \mathbf{F}\right)^{-1}$ - inverse frame operator for $\mathcal{F}$.
- $\mathbf{F}^{+}: \mathbb{R}^{M} \rightarrow \mathbb{R}^{N}$.
- $0<A \leq B<\infty$ - bounds for $\mathcal{F}$.


## Summary of Notation (contd.)

- $\widetilde{\mathbf{x}} \in \mathbb{R}^{M}$ - a representation of $\mathbf{y}$ in $\widetilde{\mathcal{F}}$.
- $\widetilde{\mathbf{f}}_{1} \ldots \widetilde{\mathbf{f}}_{N} \in \mathbb{R}^{M}$ - frame vectors for $\tilde{\mathcal{F}}$.
- $\left(\mathbf{F}^{+}\right)^{\mathrm{T}}=\left[\widetilde{\mathbf{f}}_{1}\left|\widetilde{\mathbf{f}}_{2}\right| \ldots \mid \widetilde{\mathbf{f}}_{N}\right]$ - frame operator for $\mathcal{F}$.
- $\left(\mathbf{F}^{+}\right)^{\mathrm{T}}: \mathbb{R}^{N} \rightarrow \mathbb{R}^{M}$.
- $\mathbf{F}^{\mathrm{T}}$ - inverse frame operator for $\widetilde{\mathcal{F}}$.
- $\mathbf{F}^{\mathrm{T}}: \mathbb{R}^{M} \rightarrow \mathbb{R}^{N}$.
- $0<\frac{1}{B} \leq \frac{1}{A}<\infty-$ bounds for $\widetilde{\mathcal{F}}$.

