Frames vs. Bases

- A set of vectors form a *basis* for $\mathbb{R}^M$ if they span $\mathbb{R}^M$ and are linearly independent.

- A set of $N \geq M$ vectors form a *frame* for $\mathbb{R}^M$ if they span $\mathbb{R}^M$. 
Basis Matrix

Let $\mathcal{B}$ consist of the $M$ basis vectors, $b_1 \ldots b_N \in \mathbb{R}^M$. Let $x \in \mathbb{R}^M$ be a representation of $y \in \mathbb{R}^M$ in $\mathcal{B}$. It follows that

$$y = x_1 b_1 + x_2 b_2 + \cdots + x_M b_M.$$  

This is just the matrix vector product

$$y = Bx$$

where the basis matrix, $B$, is the $M \times M$ matrix,

$$B = \begin{bmatrix} b_1 & b_2 & \cdots & b_M \end{bmatrix}.$$
Inverse Basis Matrix

To find the representation of the vector $\mathbf{x}$ in the basis $\mathcal{B}$ we multiply $\mathbf{y}$ by $\mathbf{B}^{-1}$:

$$
\mathbf{x} = \mathbf{B}^{-1} \mathbf{y}.
$$

The components of the representation of $\mathbf{x}$ in $\mathcal{B}$ are inner products of $\mathbf{y}$ with the rows of $\mathbf{B}^{-1}$. The transposes of these row vectors form a dual basis $\mathcal{\tilde{B}}$. 
Figure 1: Primal $\mathcal{B}$ (right) and dual $\tilde{\mathcal{B}}$ (left) bases and standard basis (center). The vectors which comprise $\tilde{\mathcal{B}}$ are the transposes of the rows of $\mathbf{B}^{-1}$. 
Frame Matrix

Let $\mathcal{F}$ consist of the $N$ frame vectors, $f_1 \ldots f_N \in \mathbb{R}^M$, where $N \geq M$. Let $x \in \mathbb{R}^N$ be a representation of $y \in \mathbb{R}^M$ in $\mathcal{F}$. It follows that

$$y = x_1 f_1 + x_2 f_2 + \cdots + x_N f_N.$$ 

This is just the matrix vector product

$$y = Fx$$

where the frame matrix, $F$, is the $M \times N$ matrix,

$$F = \begin{bmatrix} f_1 & f_2 & \cdots & f_N \end{bmatrix}.$$
Inverse Frame Matrix (contd.)

We might guess that

\[ x = F^{-1}y \]

where \( FF^{-1} = I \). Unfortunately, because \( F \) is not square, it has no simple inverse. However, it has an infinite number of right-inverses. Each of the \( x \) produced when \( y \) is multiplied by a distinct right-inverse is a distinct representation of the vector \( y \) in the frame, \( \mathcal{F} \).
Pseudoinverse

We observe that the pseudoinverse

$$F^+ = F^T (FF^T)^{-1}$$

is a right-inverse of $F$. We call the $N \times M$ matrix, $F^+$, an inverse frame matrix because it maps vectors, $y \in \mathbb{R}^M$, into representations, $x \in \mathbb{R}^N$. 
Frame Bounds

Let $\mathcal{F}$ consist of the $N$ frame vectors, $f_1 \ldots f_N \in \mathbb{R}^M$, where $N \geq M$, and let $\mathbf{F}^+$ be the inverse frame matrix. $\mathcal{F}$ is a frame iff for all $\mathbf{y} \in \mathbb{R}^M$ there exist $A$ and $B$ where $0 < A \leq B < \infty$ and where

$$\frac{1}{B} \| \mathbf{y} \|^2 \leq \| \mathbf{F}^+ \mathbf{y} \|^2 \leq \frac{1}{A} \| \mathbf{y} \|^2.$$ 

$A$ and $B$ are called the frame bounds.
Dual Frame

If $\mathcal{F}$ consists of the $N$ frame vectors, $f_1 \ldots f_N \in \mathbb{R}^M$, with inverse frame matrix $F^+$, then the dual frame, $\tilde{\mathcal{F}}$, consists of the $N$ frame vectors, $\tilde{f}_1 \ldots \tilde{f}_N \in \mathbb{R}^M$:

$$(F^+)^T = \begin{bmatrix} \tilde{f}_1 & \tilde{f}_2 & \ldots & \tilde{f}_N \end{bmatrix}.$$  

Let $\tilde{x} \in \mathbb{R}^N$ be a representation of $y \in \mathbb{R}^M$ in $\tilde{\mathcal{F}}$. It follows that

$$y = (F^+)^T \tilde{x}.$$  

Consequently, $(F^+)^T$ is the frame matrix for the dual frame, $\tilde{\mathcal{F}}$. 
Figure 2: Primal $\mathcal{F}$ (right) and dual $\tilde{\mathcal{F}}$ (left) frames and standard basis (center). The vectors which comprise $\tilde{\mathcal{F}}$ are the transposes of the rows of $F^+$. 
Dual Frame (contd.)

Because $F^T$ is a right inverse of $(F^+)^T$:

$$(F^+)^T F^T = I.$$ 

It follows that $F^T$ is the inverse frame matrix for the dual frame, $\widetilde{F}$, and

$$A||y||^2 \leq ||F^Ty||^2 \leq B||y||^2.$$ 

for all $y \in \mathbb{R}^M$. 
Example

What is the representation of $y = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ in the frame formed by the vectors $f_1 = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}^T$, $f_1 = \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}^T$ and $f_3 = \begin{bmatrix} 0 & -1 \end{bmatrix}^T$?

$$F = \begin{bmatrix} 0.70711 & -0.70711 & 0 \\ 0.70711 & 0.70711 & -1 \end{bmatrix}$$

$$F^+ = \begin{bmatrix} 0.70711 & 0.35355 \\ -0.70711 & 0.35355 \\ 0 & -0.5 \end{bmatrix}$$

$$F^+ y = \begin{bmatrix} 1.06066 \\ -0.35355 \\ -0.5 \end{bmatrix}$$
Tight-Frames

If $A = B$ then

$$||F^T y||^2 = A||y||^2$$

and $F$ is said to be a tight-frame. When $F$ is a tight-frame,

$$F^+ = \frac{1}{A}F^T.$$ 

If $||f_i|| = 1$ for all frame vectors, $f_i$, then $A$ equals the overcompleteness of the representation. When $A = B = 1$, then $F$ is an orthonormal basis and $F = \tilde{F}$. 
Figure 3: Primal $\mathcal{F}$ (right) and dual $\tilde{\mathcal{F}}$ (left) tight-frames with overcompleteness two and standard basis (center).
Example

What is the representation of $y = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ in the frame formed by the vectors $f_1 = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$, $f_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$, $f_3 = \begin{bmatrix} 0 & -1 \end{bmatrix}^T$ and $f_4 = \begin{bmatrix} -1 & 0 \end{bmatrix}^T$?

$F = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix}$

$F^+ = \frac{1}{2} F^T = \begin{bmatrix} 0 & 0.5 \\ 0.5 & 0 \\ 0 & -0.5 \\ -0.5 & 0 \end{bmatrix}$

$\frac{1}{2} F^T y = \begin{bmatrix} 0.5 \\ 0.5 \\ -0.5 \\ -0.5 \end{bmatrix}$
Figure 4: Primal $\mathcal{F}$ (right) and dual $\tilde{\mathcal{F}}$ (left) tight-frames with overcompleteness one (orthonormal bases) and standard basis (center).
Summary of Notation

- \( y \in \mathbb{R}^M \) – a vector.
- \( x \in \mathbb{R}^N \) – a representation of \( y \) in \( \mathcal{F} \).
- \( f_1 \ldots f_N \in \mathbb{R}^M \) where \( N \geq M \) – frame vectors for \( \mathcal{F} \).
- \( \mathbf{F} = \begin{bmatrix} f_1 & f_2 & \ldots & f_N \end{bmatrix} \) – frame matrix for \( \mathcal{F} \).
- \( \mathbf{F} : \mathbb{R}^N \rightarrow \mathbb{R}^M \).
- \( \mathbf{F}^+ = \mathbf{F}^T \left( \mathbf{F}^T \mathbf{F} \right)^{-1} \) – inverse frame matrix for \( \mathcal{F} \).
- \( \mathbf{F}^+ : \mathbb{R}^M \rightarrow \mathbb{R}^N \).
- \( 0 < A \leq B < \infty \) – bounds for \( \mathcal{F} \).
Summary of Notation (contd.)

• \( \tilde{x} \in \mathbb{R}^M \) – a representation of \( y \) in \( \tilde{F} \).
• \( \tilde{f}_1 \ldots \tilde{f}_N \in \mathbb{R}^M \) – frame vectors for \( \tilde{F} \).
• \( (F^+)^T = \begin{bmatrix} \tilde{f}_1 & \tilde{f}_2 & \ldots & \tilde{f}_N \end{bmatrix} \) – frame matrix for \( \tilde{F} \).
• \( (F^+)^T : \mathbb{R}^N \to \mathbb{R}^M \).
• \( F^T \) – inverse frame matrix for \( \tilde{F} \).
• \( F^T : \mathbb{R}^M \to \mathbb{R}^N \).
• \( 0 < \frac{1}{B} \leq \frac{1}{A} < \infty \) – bounds for \( \tilde{F} \).