The 2D Fourier Transform

The analysis and synthesis formulas for the 2D continuous Fourier transform are as follows:

- **Analysis**
  \[ F(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-j2\pi(ux+vy)} \, dx \, dy \]

- **Synthesis**
  \[ f(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u, v) e^{j2\pi(ux+vy)} \, du \, dv \]
Separability of 2D Fourier Transform

The 2D analysis formula can be written as a 1D analysis in the $x$ direction followed by a 1D analysis in the $y$ direction:

$$F(u, v) = \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} f(x, y) e^{-j2\pi ux} \, dx \right] e^{-j2\pi vy} \, dy.$$  

The 2D synthesis formula can be written as a 1D synthesis in the $x$ direction followed by a 1D synthesis in $y$ direction:

$$f(x, y) = \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} F(u, v) e^{j2\pi ux} \, du \right] e^{j2\pi vy} \, dv.$$
Separability Theorem

\[ f(x, y) = f(x)g(y) \xrightarrow{\mathcal{F}} F(u, v) = F(u)G(v) \]

Proof:

\[
F(u, v)
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-j2\pi(ux+vy)} \, dx \, dy
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x)g(y) e^{-j2\pi ux} e^{-j2\pi vy} \, dx \, dy
= \int_{-\infty}^{\infty} f(x) e^{-j2\pi ux} \, dx \int_{-\infty}^{\infty} g(y) e^{-j2\pi vy} \, dy
= F(u)G(v)
\]
The 2D Discrete Fourier Transform

The analysis and synthesis formulas for the 2D discrete Fourier transform are as follows:

- **Analysis**

  \[
  \hat{F}(k, \ell) = \frac{1}{\sqrt{MN}} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} F(m, n) e^{-j2\pi \left(\frac{km}{M} + \frac{\ell n}{N}\right)}
  \]

- **Synthesis**

  \[
  F(m, n) = \frac{1}{\sqrt{MN}} \sum_{k=0}^{M-1} \sum_{\ell=0}^{N-1} \hat{F}(k, \ell) e^{j2\pi \left(\frac{km}{M} + \frac{\ell n}{N}\right)}
  \]
Separability of the 2D DFT

\[ \hat{F}(k, \ell) = \]
\[ \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \left[ \frac{1}{\sqrt{M}} \sum_{m=0}^{M-1} F(m, n) e^{-j2\pi \frac{k m}{M}} \right] e^{-j2\pi \frac{\ell n}{N}} \]

The 2D forward DFT can be written in matrix notation:

\[ \hat{F} = (W^* F) W^* \]

where

\[ W^*_{mn} = \frac{1}{\sqrt{N}} e^{-j2\pi m n} \]

and

\[ F(m, n) = \]
\[ \frac{1}{\sqrt{N}} \sum_{\ell=0}^{N-1} \left[ \frac{1}{\sqrt{M}} \sum_{k=0}^{M-1} \hat{F}(k, \ell) e^{j2\pi \frac{k m}{M}} \right] e^{j2\pi \frac{\ell n}{N}}. \]
Separability of the 2D DFT (contd.)

The 2D inverse DFT can be written in matrix notation:

\[ F = (W\hat{F})W \]

where

\[ W_{mn} = \frac{1}{\sqrt{N}} e^{j2\pi m\frac{n}{N}}. \]