Functions of Random Variables

Let’s consider a continuous r.v., $Y$, which is a differentiable, increasing function of a second continuous r.v., $X$:

$$Y = g(X).$$

Because $g$ is differentiable and increasing, $g'$ and $g^{-1}$ are guaranteed to exist. Because $g$ maps all $x \leq s \leq x + \Delta x$ to $y \leq t \leq y + \Delta y$:

$$\int_{x}^{x+\Delta x} f_X(s) ds = \int_{y}^{y+\Delta y} f_Y(t) dt.$$

It follows that for small $\Delta x$:

$$f_Y(y)\Delta y \approx f_X(x)\Delta x.$$
Functions of Random Variables (contd.)

Dividing by $\Delta y$, we get an approximate expression for $f_Y$ in terms of $f_X$:

$$f_Y(y) \approx f_X(x) \frac{\Delta x}{\Delta y}.$$
Functions of Random Variables (contd.)

This expression is exact in the limit $\Delta x \to 0$:

$$f_Y(y) = \lim_{\Delta x \to 0} f_X(x) \frac{\Delta x}{\Delta y}$$

$$= \lim_{\Delta x \to 0} f_X(x) \frac{1}{\Delta y/\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f_X(x)}{\Delta y/\Delta x}.$$

From calculus we know that:

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{g(x + \Delta x) - g(x)}{\Delta x} = g'(x).$$

Consequently

$$f_Y(y) = \frac{f_X(x)}{g'(x)}.$$
Functions of Random Variables (contd.)

Substituting $g^{-1}(y)$ for $x$:

$$f_Y(y) = \frac{f_X(g^{-1}(y))}{g'(g^{-1}(y))}$$

yields an expression for $f_Y$ in terms of $g'$, $g^{-1}$, and $f_X$. 
Linear Example

Consider a continuous r.v., $Y$, which is a linear function of a continuous r.v., $X$. Specifically, $Y = aX + b$. It follows that

$$
g(x) = ax + b$$
$$g'(x) = a$$
$$g^{-1}(y) = (y - b)/a.
$$

Substituting the above into

$$f_Y(y) = \frac{f_X(g^{-1}(y))}{g'(g^{-1}(y))}$$

yields

$$f_Y(y) = \frac{1}{a}f_X\left(\frac{y - b}{a}\right).$$
Linear Example (contd.)

Let \( f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-(x-\mu)^2/2} \), then

\[
f_Y(y) = \frac{1}{a\sqrt{2\pi}} e^{-\left(\frac{y-b}{a} - \mu\right)^2/2}.
\]
Linear Example (contd.)

Unfortunately, there is a problem. When $a = -1$, the function $g$ is not increasing (it is decreasing). Consequently,

$$-f_Y(y) \Delta y \approx f_X(x) \Delta x.$$ 

It follows that for decreasing functions,

$$f_Y(y) = \frac{f_X(g^{-1}(y))}{-g'(g^{-1}(y))}.$$ 

However, we can derive a function which is correct in both cases by replacing $g'(.)$ with $|g'(.)|$ in the expression relating $f_Y(.)$ and $f_X(.)$:

$$f_Y(y) = \frac{f_X(g^{-1}(y))}{|g'(g^{-1}(y))|}.$$
Quadratic Example

Consider a continuous r.v., $Y$, which is a quadratic function of a continuous r.v., $X$. Specifically, $Y = X^2$. It follows that

\[
\begin{align*}
g(x) &= x^2 \\
g'(x) &= 2x \\
g^{-1}(y) &= \sqrt{y}.
\end{align*}
\]

Substituting the above into

\[
f_Y(y) = \frac{f_X(g^{-1}(y))}{|g'(g^{-1}(y))|}
\]

yields

\[
f_Y(y) = \frac{f_X(\sqrt{y})}{2\sqrt{y}}.
\]
Quadratic Example (contd.)

Let $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, then

$$f_Y(y) = \frac{e^{-y/2}}{\sqrt{2\pi}|2\sqrt{y}|}.$$ 

Unfortunately, there is a problem:

$$\int_{0}^{\infty} f_Y(y) dy = \int_{0}^{\infty} \frac{e^{-y/2}}{\sqrt{2\pi}|2\sqrt{y}|} dy = \frac{1}{2} \neq 1.$$ 

Can anyone see the mistake?
Quadratic Example (contd.)

The mistake is that two different values of $Y$ satisfy $Y = X^2$:

$$g^{-1}(y) = \pm \sqrt{y}.$$

We decided to use the positive square root arbitrarily and ignored the negative square root. Hence the factor of two error. In general, if a function does not have a unique inverse, we must sum over all possible inverse values:

$$f_Y(y) = \sum_{i=1}^{n} \frac{f_X(g^{-1}_i(y))}{|g'(g^{-1}_i(y))|}.$$
Quadratic Example (contd.)

Let \( g_1^{-1}(y) = \sqrt{y} \) and \( g_2^{-1}(y) = -\sqrt{y} \), then

\[
f_Y(y) = \frac{f_X(g_1^{-1}(y))}{|g'(g_1^{-1}(y))|} + \frac{f_X(g_2^{-1}(y))}{|g'(g_2^{-1}(y))|}
= e^{-y/2} \sqrt{2\pi y}.
\]

The above p.d.f. defines a distribution called the *chi square* distribution.
Kinetic Energy

Recall from elementary physics, that the kinetic energy, $K$, of a moving particle is given by

$$K = \frac{1}{2}mV^2$$

where $m$ is mass and $V$ is velocity. Let $V$ be a normally distributed random variable with mean, $\mu$, and variance, $\sigma^2$:

$$f_V(v) = \frac{1}{\sigma \sqrt{2\pi}}e^{-\frac{(v-\mu)^2}{2\sigma^2}}.$$

We would like to compute $f_K(k)$, the p.d.f for the continuous random variable, $K$. 
Kinetic Energy (contd.)

We start by computing, $g'$, $g_1^{-1}$, and $g_2^{-1}$:

\[
g(v) = \frac{1}{2}mv^2
\]

\[
g'(v) = mv
\]

\[
g_1^{-1}(k) = \sqrt{\frac{2k}{m}}
\]

\[
g_2^{-1}(k) = -\sqrt{\frac{2k}{m}}.
\]

Substituting $f_V(v)$ and the above into

\[
f_K(k) = \frac{f_V(g_1^{-1}(k))}{|g'(g_1^{-1}(k))|} + \frac{f_V(g_2^{-1}(k))}{|g'(g_2^{-1}(k))|}
\]

yields

\[
f_K(k) = e^{-\left(\frac{\sqrt{2k/m-\mu}^2}{2\sigma^2}\right)} + e^{-\left(-\frac{\sqrt{2k/m-\mu}^2}{2\sigma^2}\right)}\frac{\sigma}{\sigma\sqrt{\pi km}}.
\]