1. The joint probability distribution of two discrete random variables $X$ and $Y$ is given below:

$$
\begin{bmatrix}
Y \\
X \\
\end{bmatrix} = 
\begin{bmatrix}
0 & 1/9 & 1/3 \\
1/12 & 5/36 & 1/18 \\
1/36 & 1/4 & 0 \\
\end{bmatrix}.
$$

Compute:

(a) Marginal information, $H_X$.
(b) Marginal information, $H_Y$.
(c) Conditional information, $H_{X|Y}$.
(d) Conditional information, $H_{Y|X}$.
(e) Joint information, $H_{XY}$.
(f) Mutual information, $I_{XY}$.

2. A fair coin is flipped until the first head appears. Let $X$ denote the number of flips required. Find the entropy, $H_X$, in bits. The following expressions may be useful: $\sum_{n=1}^{\infty} r^n = r/(1 - r)$ and $\sum_{n=1}^{\infty} nr^n = r/(1 - r)^2$.

3. Two discrete random variables, $X$ and $Y$, have outcomes, \{x_1, x_2, x_3\} and \{y_1, y_2, y_3\}, which occur with probabilities, $p_X(x_i) = \{1/2, 1/4, 1/4\}$ and $p_Y(y_j) = \{2/3, 1/6, 1/6\}$. Compute the maximum entropy joint distribution, $P_{XY}$, with these marginals. Prove your result is correct.

4. The p.m.f. for the 12367 most frequently used words in English is approximately:

$$
p(n) = \begin{cases}
\frac{0.1}{n} & \text{for } 1 \leq n \leq 12367 \\
0 & n > 12367.
\end{cases}
$$

This remarkable fact is known as Zipf's law, and applies to many languages (Zipf, 1949). If we assume that English is generated by picking words at random according to this distribution, what is the entropy of English (per word)?
5. Write a MATLAB function which given a pair of images, \( X \) and \( Y \), of equal size, computes the mutual information, \( I_{XY} \). Using the Io image, compute \( I_{RG} \), \( I_{RB} \), and \( I_{GB} \).

6. Write a MATLAB function, \( cdf \), which given an image with grey values, \( k \), in the range 0 to 255, returns a vector of length 256 representing the discrete c.d.f., \( F_K(k) = \sum_{i=0}^{k} f_K(k) \), where \( f_K(k) \) is the image’s histogram. The output of \( cdf \) should be normalized so that its minimum value is 0 and its maximum value is 255. Test your function on the Ganymede image. Plot your result.

7. Write a MATLAB function, \( icdf \), which given an image with grey values in the range 0 to 255, returns a vector of length 256 representing the inverse function of the discrete c.d.f., \( F_K \), defined as above. The output of \( icdf \) should be normalized so that its minimum value is 0 and its maximum value is 255. Test your function on the Callisto image. Plot your result.

8. Using \( cdf \) and \( icdf \), write a MATLAB function, \( match \), which given two images, \( I_1 \) and \( I_2 \), returns a new image, \( I_3 \). The value of every pixel, \((i, j)\), in \( I_3 \) is computed as follows:

\[
I_3(i, j) = F_2^{-1}(F_1(I_1(i, j)))
\]

where \( F_1 \) is the c.d.f. of \( I_1 \) and \( F_2^{-1} \) is the inverse c.d.f. of \( I_2 \). Compute \( I_3 \) where \( I_1 \) is Ganymede and \( I_2 \) is Callisto. Display \( I_3 \). Plot the histograms of \( I_2 \) and \( I_3 \).

9. JPEG\(^1\) is by far the most widely used compressed image format. However, unlike the GIF format, which uses a lossless compression method, JPEG compression decreases image quality, i.e., it is a lossy method. In this exercise, JPEG will be viewed as an information channel. The grey values of the pixels of an image before and after JPEG compression will be considered to be samples of two non-independent discrete r.v.’s \( X \) and \( Y \). Note that a pixel of an uncompressed image with 256 grey levels can contain at most 8 bits of information. The lena image on the class homepage is stored in a PGM format. This format does no compression. The lena-jpeg image is also stored in the PGM format.

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\(^1\)Joint Photograph Experts Group.
format. However, the lena-jpeg image has already undergone JPEG compression. Using pixels of the lena and lena-jpeg images, compute the following:

- \( H_X \) - The entropy of the lena image.
- \( H_Y \) - The entropy of the lena-jpeg image.
- \( H_{Y|X} \) - The channel noise.
- \( H_{X|Y} \) - The channel loss.
- \( I_{XY} \) - The mutual information, \( i.e.\), the amount of information which actually passes through the JPEG information channel.