

## CS 530: Geometric and Probabilistic Methods in Computer Science Homework 4 (Fall '08)

1. Characterize the following as linear, or non-linear, and as even, odd, or neither, and prove your characterization.

- (a)  $f(x) = -x$
- (b)  $f(x) = 7x + 1$
- (c)  $f(x) = x^2 - x$
- (d)  $f(x) = e^{-x^2}$
- (e)  $f(x) = \cos x$ .

2. Write a function in MATLAB which, given an  $N \times 1$  vector,  $\mathbf{x}$ , will return an  $N \times N$  circulant matrix:

$$\mathbf{A} = \left[ \begin{array}{c|c|c|c} \mathbf{S}^0 \mathbf{x} & \mathbf{S}^1 \mathbf{x} & \dots & \mathbf{S}^{N-1} \mathbf{x} \end{array} \right]$$

where  $S_{ij}^n = \delta(i - j - n \bmod N)$  and  $\delta$  is the Kronecker delta function.

3. Using MATLAB compute the following:

- (a) A  $4 \times 4$  circulant matrix  $\mathbf{A}$ , where the first column of  $\mathbf{A}$  is  $\mathbf{x} = [-1 \ 2 \ -3 \ 4]^T$ .
- (b)  $\mathbf{W}$ , the matrix of right eigenvectors of  $\mathbf{A}$ , and  $\Lambda$ , the diagonal matrix of eigenvalues.
- (c)  $\mathbf{W}^{-1}$  and  $(\mathbf{W}^*)^T$ .
- (d)  $\mathbf{A}\mathbf{y}$  where  $\mathbf{y} = [1 \ 2 \ 3 \ 4]^T$ .
- (e)  $\mathbf{W}\Lambda(\mathbf{W}^*)^T\mathbf{y}$  where  $\mathbf{y} = [1 \ 2 \ 3 \ 4]^T$ .
- (f) A  $100 \times 100$  circulant matrix  $\mathbf{B}$  where the first column of  $\mathbf{B}$  is  $\mathbf{x} = [-1 \ 2 \ -3 \ \dots \ 100]^T$ . There is no need to print it out.
- (g) Let  $\lambda$  be the eigenvalue of  $\mathbf{B}$  with twelfth smallest magnitude. Plot the left and right eigenvectors of  $\mathbf{B}$  with eigenvalue equal to  $\lambda$ . Note: The eigenvectors are complex. The real and imaginary parts should be plotted as functions of time.

4. Prove that  $\sin(x) = \frac{e^{jx} - e^{-jx}}{2j}$ .

5. The impulse response function of a linear shift-invariant system is:

$$h(t) = \frac{\sin(\pi t)}{\pi t}$$

and its input is:

$$x(t) = \cos(4\pi t) + \cos(\pi t/2).$$

What is its output?

6. The impulse response function of a linear shift-invariant system is:

$$h(t) = e^{-\frac{\pi t^2}{2}}$$

and its input is:

$$x(t) = e^{j2\pi s_0 t}.$$

What is its output?

7. The sine Gabor function is the product of a sine and a Gaussian,  $f(t) = e^{-\pi t^2} \sin(2\pi s_0 t)$ . Give an expression for  $F(s)$ , the Fourier transform of  $f(t)$ .

8. Prove that the sum of two independent Gaussian random variables with zero mean and variances  $\sigma_1^2$  and  $\sigma_2^2$  is a Gaussian random variable with zero mean and variance  $\sigma_1^2 + \sigma_2^2$ .

9. The function,  $f(t)$ , is defined as:

$$f(t) = \begin{cases} 1 & \text{if } |at - b| \leq \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases}$$

Give an expression for  $F(s)$ , the Fourier transform of  $f(t)$ .

10. The transfer function of a linear shift invariant system is  $H(s) = 1/s$ . The impulse response function,  $h(t)$ , is  $\mathcal{F}^{-1}\{H(s)\}$ . Give an expression for  $g(t)$  where:

$$g(t) = \int_{-\infty}^{\infty} e^{j2\pi s_0 \tau} h(t - \tau) d\tau.$$

11. Compute the Fourier transform of  $f(t) = -2\pi t e^{-\pi t^2} \cos(2\pi s_0 t)$ . Hint: What is  $\frac{d(e^{-\pi t^2})}{dt}$ ?

12. Prove the following statement: If  $\mathcal{F}\{f\}(s) = F(s)$  then  $\mathcal{F}\{F\}(s) = f(-s)$ .  
Hint: If  $\mathcal{F}\{f\}(s) = F(s)$  then  $\mathcal{F}^{-1}\{F\}(t) = f(t)$ .
13. Prove that  $\mathcal{F}^{-1}\{\mathcal{F}\{f\}\} = f$
14. Write a MATLAB function which takes an integer argument,  $N$ , and computes the truncated Fourier series:

$$f_N(t) = \pi + \sum_{\substack{\omega = -N \\ \omega \neq 0}}^N \frac{j}{\omega} e^{j\omega t}$$

Plot the real and imaginary parts of  $f_N(t)$  for  $N = 1, 3, 6, 12, 24$  and  $48$ . Plot the functions on the interval,  $-4\pi \leq t \leq 4\pi$ .