CS 530: Geometric and Probabilistic Methods in Computer Science
Homework 5 (Fall '08)

1. Consider the following *advection-diffusion* partial differential equation:

$$\frac{\partial P}{\partial t} = -C\frac{\partial P}{\partial x} + D\frac{\partial^2 P}{\partial x^2}.$$

   (a) Give finite difference approximations for $\frac{\partial P}{\partial t}|_{x,t}$, $\frac{\partial P}{\partial x}|_{x,t}$, and $\frac{\partial^2 P}{\partial x^2}|_{x,t}$.

   (b) Give an expression for $P(x,t+\Delta t)$ in terms of $P(x,t)$, $P(x+\Delta x,t)$, and $P(x-\Delta x,t)$.

2. A bivariate Gaussian random variable, $\mathbf{x} = \begin{bmatrix} x_0 & x_1 \end{bmatrix}^{\mathrm{T}}$, has the following p.d.f.:

$$f(x_0,x_1) = \frac{(a^2-b^2)^{\frac{1}{2}}}{2\pi}\exp\left(-\frac{1}{2}\left[ax_0x_0 + 2bx_0x_1 + ax_1x_1\right]\right).$$

   (a) Give the matrix $\mathbf{W}$ which will decorrelate the components of $\mathbf{x}$.

   (b) Let $\mathbf{u} = \mathbf{W}\mathbf{x}$. Give an expression for $g(u_0,u_1)$, the p.d.f. for the bivariate Gaussian random variable, $\mathbf{u} = \begin{bmatrix} u_0 & u_1 \end{bmatrix}^{\mathrm{T}}$.

3. Let $S_r(\lambda)$, $S_g(\lambda)$, and $S_b(\lambda)$ be the spectral sensitivity functions of the red, green, and blue cones of the human retina. Let $C(\lambda)$ be the spectral distribution of a sunlit daffodil. Define a system of linear equations, which when solved, gives the amounts, $V_r(C)$, $V_g(C)$, and $V_b(C)$, of the three CIE standard primary sources, $\delta(\lambda-700)$, $\delta(\lambda-546)$, and $\delta(\lambda-436)$, necessary to reproduce the color of the sunlit daffodil.

4. The n-th moment of Ψ is defined to be $M_n\{\Psi\} = \int_{-\infty}^{\infty} t^n\Psi(t)dt$. Let $f(t) = e^{-\pi t^2}$, $f'(t) = -2\pi t e^{-\pi t^2}$, and $f''(t) = 2\pi e^{-\pi t^2}(2\pi t^2 - 1)$. Prove the following:

   (a) $M_0\{f'\} = 0$.

   (b) $M_0\{f''\} = M_1\{f''\} = 0$.

5. The six vectors, $\mathbf{f}_1 = \begin{bmatrix} \cos(\pi/3) & \sin(\pi/3) \end{bmatrix}^{\mathrm{T}}$, $\mathbf{f}_2 = \begin{bmatrix} \cos(\pi/3) & -\sin(\pi/3) \end{bmatrix}^{\mathrm{T}}$, $\mathbf{f}_3 = \begin{bmatrix} -1 & 0 \end{bmatrix}^{\mathrm{T}}$, $\mathbf{f}_4 = \begin{bmatrix} -\cos(\pi/3) & -\sin(\pi/3) \end{bmatrix}^{\mathrm{T}}$, $\mathbf{f}_5 = \begin{bmatrix} -\cos(\pi/3) & \sin(\pi/3) \end{bmatrix}^{\mathrm{T}}$, and $\mathbf{f}_6 = \begin{bmatrix} 1 & 0 \end{bmatrix}^{\mathrm{T}}$ form a frame $\mathcal{F}$ for $\mathbb{R}^2$. Draw the frame.

(a) Give two representations for the vector, $\mathbf{x} = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$, in $\mathcal{F}$

(b) Prove that $\mathbf{x}$ has an infinite number of representations in $\mathcal{F}$.

(c) Give a matrix which transforms any representation of a vector in $\mathcal{F}$ into its representation in the standard basis for $\mathbb{R}^2$.

(d) Give a matrix which transforms a representation of any vector in the standard basis for $\mathbb{R}^2$ into its representation in $\mathcal{F}$.